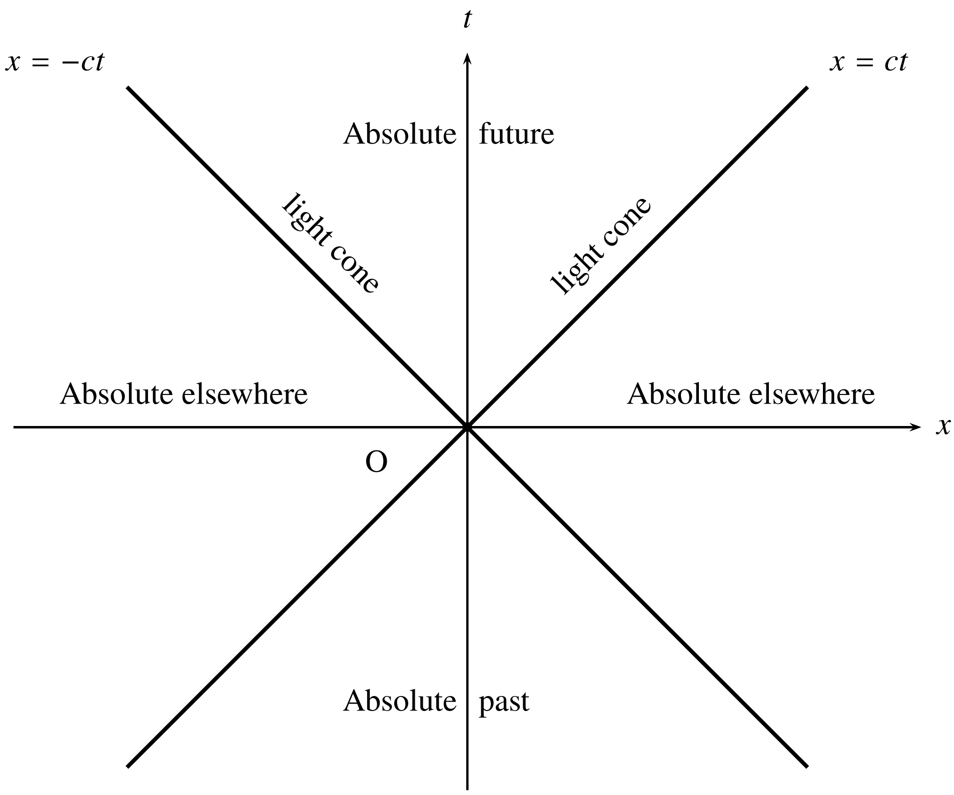
THE RELATIVITIES



THE NOTEBOOKS OF FONTENAY

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In the context of the continuing education days organized each year in September by the Physics and Chemistry Sections of the School, Mr. Lévy-Leblond, Professor at the University of Paris VII, very kindly agreed in 1970 to give some lectures on the problems posed by the teaching of relativity theories. In such a short time, it took all the art and presence of an excellent professor to convey this scientific discourse so rich in extensions. It was attentive listeners who attempted to transcribe these lectures without losing, due to the written form, this vivid language, these aside reflections, which allow a speaker to make an audience feel the essential without insisting on details, particularly on calculations. This new, often very original approach can encourage the reading of more specialized works, or help synthesize pre-acquired knowledge. The fruit of team work bringing together the teachers of the School's Physics Section and secondary school teachers, it can be read not only by physics professors but also by mathematics professors, philosophy professors and all those interested in the great currents of contemporary thought and thus help some to step out of the framework of their own discipline.

That Mr. Lévy-Leblond who agreed to reread and correct the manuscript find here the expression of our thanks.

H. Delavault

Necessity of a overhaul of the modes of presentation of modern physics

The teaching of a scientific discipline traditionally relies on the chronological evolution of the science in question; it is therefore an a-critical mode of presentation. However, between the moment a science is born and when it is taught, 50 or 100 years later, conceptions evolve. It is then necessary to overhaul the teaching to adopt a *logical* order rather than a *chronological* one, relying on the daily practice of the science in question. This "reworking" must be constant: such an attitude is obviously more demanding, but surely more interesting and fruitful than the traditional behavior. It turns out that this process of overhauling the body of doctrine was automatic for most theories of classical physics, including electromagnetism: thus, mechanics is obviously no longer taught in the terms in which Newton conceived it! But this permanent overhaul has not been accomplished for "modern" physics (relativity and quantum mechanics) which continues to be taught and thought for the essential in the terms in which it was formulated 60 or 70 years ago. The causes of this blockage, this delay, are manifest: they reside in the industrialization of scientific production. The division of labor hinders reflection on the content of science:

1) horizontal division
   * between specialties that do not communicate with each other.
   * between teaching and research.
2) vertical division between those who define research paths (but do not research) and researchers (from whom it is not expected that they question the science they practice).

Scientific knowledge accumulates, certainly, but without being mastered and conceptions of science tend to freeze.

Contents

Chapter 1

RELATIVITY AND RELATIVITIES

(written by Madeleine Sonneville)

Relativity, too often attributed to Einstein, does not exist...

What exists, on the other hand, is THE *principle of relativity*: we know Galileo's formulation, in 1632, in the "Dialogue Concerning the Two Chief World Systems". This principle had already been expressed, in almost similar terms, by Chinese scholars, fifteen centuries before Galileo.

To this principle of relativity, which has a long history, we oppose THE *theories of relativity*.

1) one of them is "Galilean" relativity
2) another will be "Einsteinian" relativity formulated in 1905

1.1 The principle of relativity

1.1.1 Statement

A very simple statement of the principle of relativity is, for example, the following:

There exist, on the physical world, equivalent points of view.

That is to say that different observation conditions lead, in some cases, to the same description of the physical world. It can be stated more formally:

There exist reference frames (reference systems in space-time) equivalent for the laws of physics, that is, such that the laws of physics have the same form in two of these reference frames.

It is understood that a physical law in a reference frame is a functional relation F between the values of various physical quantities in this reference frame. So we have

$$F(a, b, c, \dots) = 0 \text{ in the reference frame } \mathscr{R}$$
$$F(a', b', c', \dots) = 0 \text{ in the reference frame } \mathscr{R}'$$

the functional relation F is the same; on the other hand, the values "a" and "a'" of the same physical quantity *may* be different in $\mathscr{R}$ and $\mathscr{R}'$: "a" and "a'" are for example the distances of a mobile from the origin of spaces in $\mathscr{R}$ and $\mathscr{R}'$ which differ by the choice of the origin of spaces. So in general:

$$a \neq a',\ b \neq b',\ c \neq c', \dots$$

but the same relation.

This affirmation of *the existence of certain classes of equivalent reference frames* therefore constitutes a *strong principle* because their existence is not obvious.

1.1.2 What are these equivalent reference frames?

1.1.2.1 The "translated" reference frames

They differ by the choice of the origin of times or spaces (it is a spatial or temporal translation), the orientations of the space axes being the same.

The equivalence of such reference frames is linked to the fact that the laws of physics involve only *distances* or *durations* and never *positions* or *instants*; the origins of times or spaces are therefore indifferent.

To verify the equivalence of such reference frames, translating the experiment in time or space may be more convenient than translating the reference frame. But it is obviously important to displace (in space for example) the entire experiment and not just a measuring device: do not forget to translate the Earth if it is an experiment on the fall of bodies!

At the limit, one must obviously translate the entire universe and the principle of relativity would then have no interest since it would affirm here the *homogeneity* of an *empty* universe.

The interest of the principle of relativity is therefore, in this case, linked to the possibility, essential in physics, of isolating "pieces of universe" more or less important, neglecting certain influences over others and considering only a suitably chosen fraction of the universe as a "physical system".

1.1.2.2 The "rotated" reference frames

They differ from each other by the orientation of the space axes. The principle of relativity therefore affirms here the properties of *isotropy* of space. These two first classes of equivalent reference frames were considered only very late due to their triviality.

1.1.2.3 The "moved" reference frames

in motion relative to each other we will avoid in this case the term "translation" which would lead to confusion. This is the class of equivalent reference frames most often talked about. The principle of relativity affirms here that there exists at least a certain category of relative movements between reference frames that do not affect the laws of physics.

But the principle of relativity remains abstract as long as the relations between a and a', b and b' etc. are not specified. This is what the theories of relativity will do, each in a different way, indicating how these quantities correspond when passing from one reference frame to another.

1.2 The theories of relativity

They indicate these transformation rules, more particularly for spatio-temporal quantities (lengths and times). They therefore give *formulas for transforming spatio-temporal coordinates (positions and instants)* when passing from a reference frame $\mathscr{R}$ to an equivalent reference frame $\mathscr{R}'$ (translated, rotated or moved relative to $\mathscr{R}$).

We will limit ourselves here to one space dimension (rotations will therefore not be considered; t will denote time and x the abscissa, in $\mathscr{R}$; t' and x' are their homologues in $\mathscr{R}'$.

1.2.1 For translated reference frames

1.2.1.1 in space

We will have transformation formulas of the type

$$\boxed{\begin{cases} x' = x - a \\ t' = t \end{cases}}$$

where a is the distance between the origin points

1.2.1.2 in time

We will similarly have

$$\begin{cases} x' = x \\ t' = t - b \end{cases}$$

where b is the time offset between the clocks

1.2.2 For reference frames in relative motion

1.2.2.1 "Galilean" theory of relativity

The "Galilean transformations" were not *written* by Galileo who did not have the necessary analytical tool and were probably not explicitly formulated before the 19th century, due to their apparent obviousness.

This Galilean relativity only gained interest after Einstein's work, when it was found that these formulas were false: one then wondered what the consequences of Galilean relativity were, how it differed from the new theory, how it was similar to it. This is a typical example of the "reworking" processes mentioned above: questions of relativity or invariance only gained importance in physics in the framework of Einsteinian relativity or in that of quantum mechanics (while invariances have always existed!) but they have also acquired, by recurrence, importance in classical physics. And it is, at present, very interesting to rethink classical mechanics and electromagnetism in light of these invariance principles. This reworking, a process somewhat "blocked" today for the reasons evoked above deserves more interest.

The Galilean transformation formulas are written

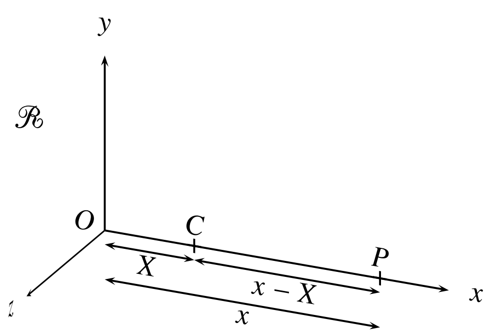
$$\begin{cases} x' = x - Vt \\ t' = t \end{cases}$$

where V is the uniform velocity of $\mathscr{R}'$ in $\mathscr{R}$.

Example 1: use of these formulas

Let Newton's law, describing the motion of a material point, be valid in a reference frame $\mathscr{R}$ linked to the Earth with origin at O:

$$f = m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$



where x is the abscissa of P counted from O, where f is the force acting on P and m the mass of P. If f is the gravitational force exerted on an apple P by the Earth with center C, it has the expression

$$\begin{aligned} f &= \frac{KMm}{(x-X)^2} \\ &= F(x-X) \end{aligned}$$

where M is the mass of the Earth, K is the universal gravitational constant, X is the abscissa of C counted from O, and F a function. In a reference frame $\mathscr{R}'$ linked to a rocket, the functional relation F defining the universal gravitational force would be the same, and the force would therefore be:

$$\begin{aligned} f' &= \frac{KMm}{(x'-X')^2} \\ &= F(x'-X') \end{aligned}$$

where x' and X' are the abscissas of the apple P and the point C counted from an origin linked to the rocket $\mathscr{R}'$ (*masses are implicitly assumed invariant*). As

$$\begin{cases} x' = x - Vt \\ X' = X - Vt \end{cases}$$

where V is the velocity of the rocket relative to the Earth. Obviously we will have

$$x - X = x' - X'$$

which translates the invariance of distances in a Galilean transformation. And so

$$f' = f$$

Not only is the functional relation F defining the force the same, but also the forces will here be invariant under Galilean transformation (this is a well-known and commonly used fact). Furthermore:

$$\begin{aligned} \frac{\mathrm{d}x'}{\mathrm{d}t'} &= \frac{\mathrm{d}x'}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(x - Vt) \\ &= \frac{\mathrm{d}x}{\mathrm{d}t} - V \end{aligned}$$

from which

$$\frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

which translate the classical law of velocity composition and *the invariance of accelerations* by a Galilean transformation. Knowing that

$$f = m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

that

$$f' = f$$

and that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x'}{\mathrm{d}t'^2}$$

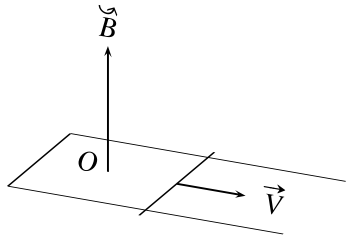
we deduce

$$f' = m\,\frac{\mathrm{d}^2 x'}{\mathrm{d}t'^2}$$

In other words, *Newton's law is also valid in $\mathscr{R}'$.*
Remark: here, *all* quantities turn out to be invariant.

<u>Example 2:</u>

Let the classic experiment on electromagnetic induction. Let's try to interpret it from a microscopic point of view, in two distinct reference frames.



In the laboratory reference frame $\mathscr{R}$, the electrons of the bar, animated by the velocity

$$\vec{v}' = \vec{V}$$

due to the motion of the bar, are subjected to the Laplace-Lorentz magnetic force

$$\vec{f} = q\vec{v} \times \vec{B}$$

hence the direction of the induced current and the calculation of the induced emf.
In the reference frame $\mathscr{R}'$ linked to the bar, the electrons are at rest and therefore undergo no magnetic force. It therefore seems that we must conclude the absence of induced current and therefore induced emf!

There is therefore a *paradox* in the sense that, in two reference frames that deduce from each other by a Galilean transformation, and which must therefore satisfy the principle of relativity, we obtain two different physical situations (existence or absence of current). But this paradox is only *apparent* because it resolves very well, even in terms of Galilean relativity.

Indeed: Write the Laplace-Lorentz law in a generalized form, taking into account the possible existence of an electric field. Then:

$$\text{in } \mathscr{R} \quad \vec{f} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$\text{in } \mathscr{R}' \quad \vec{f}' = q\left(\vec{E}' + \vec{v}' \times \vec{B}'\right)$$

Charges are assumed invariant; we have seen that forces must be invariant under a Galilean transformation, $\vec{f} = \vec{f}'$, so:

$$q\left(\vec{E} + \vec{v} \times \vec{B}\right) = q\left(\vec{E}' + \vec{v}' \times \vec{B}'\right)$$

The velocity of the charged particle not being invariant, at least one of the two fields must vary. As $\vec{v}' = \vec{v} - \vec{V}$:

$$q\left(\vec{E} + \vec{v} \times \vec{B}\right) = q\left(\vec{E}' + \vec{v} \times \vec{B}' - \vec{V} \times \vec{B}'\right)$$

and as this equality must be maintained for arbitrary $\vec{v}$, we necessarily have

$$\begin{cases} \vec{E} = \vec{E}' - \vec{V} \times \overset{\curvearrowleft}{B}' \\ \overset{\curvearrowleft}{B} = \overset{\curvearrowleft}{B}' \end{cases}$$

or, by inverting:

$$\boxed{\begin{cases} \vec{E}' = \vec{E} + \vec{V} \times \overset{\curvearrowleft}{B} \\ \overset{\curvearrowleft}{B}' = \overset{\curvearrowleft}{B} \end{cases}}$$

By a Galilean transformation, *the fields $\vec{E}$ and $\overset{\curvearrowleft}{B}$ are therefore not invariant.*
And it is therefore false to believe that the transformation of fields is a consequence of Einsteinian relativity. Nevertheless, in Einsteinian relativity, the transformation formulas will be more complex. *The interconnection of electric and magnetic fields exists in Galilean relativity.*

Let us return then to the experiment described above. Then $\vec{v}' = 0$ (the electrons are at rest relative to the bar), and $\vec{v} = \vec{V}$. We will therefore have

$$\begin{cases} \vec{E}' = \vec{v} \times \overset{\curvearrowleft}{B} \\ \overset{\curvearrowleft}{B}' = \overset{\curvearrowleft}{B} \end{cases}$$

because $\vec{E} = 0$. For the observer linked to the bar, the force is

$$\begin{aligned} \vec{f}' &= q(\vec{E}' + \vec{v}' \times \overset{\curvearrowleft}{B}') \\ &= q\vec{E}' \\ &= q\vec{v} \times \overset{\curvearrowleft}{B} \end{aligned}$$

The two observers therefore agree that

$$\vec{f} = \vec{f}'$$

What changes is the *point of view*, *the perspective*: the force appears magnetic in $\mathcal{R}$ and electric in $\mathcal{R}'$; but its value is the same. This proves that the important thing here is not the description (electric or magnetic) given of the phenomena, but the value of the force on which the two observers effectively manage to agree.

It is important to understand all the effects of relativity in terms of *perspective*: *the objects*, that is, here the laws of physics *are the same*; only *their appearance*, or the description given of them, differs according to the reference frame.

It will be the same later regarding the effects of Einsteinian relativity (dilation of durations and contraction of lengths): these are only perspective effects.

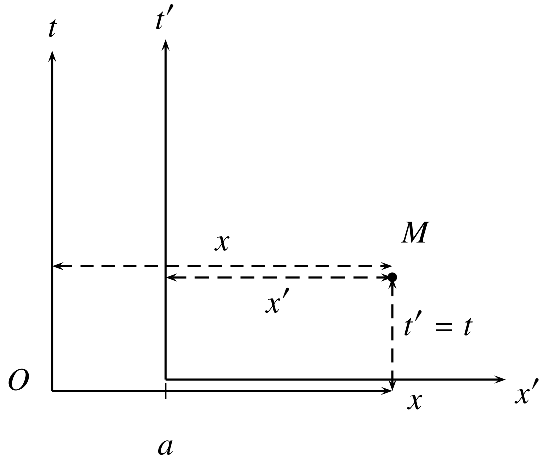
1.2.2.2 Graphical interpretation of the Galilean theory

The set of events (a place, an instant) is represented on a two-dimensional diagram. In such a representation, a change of reference frame will be a change of axes.

Thus, for changes of reference frame that leave the laws of physics invariant:

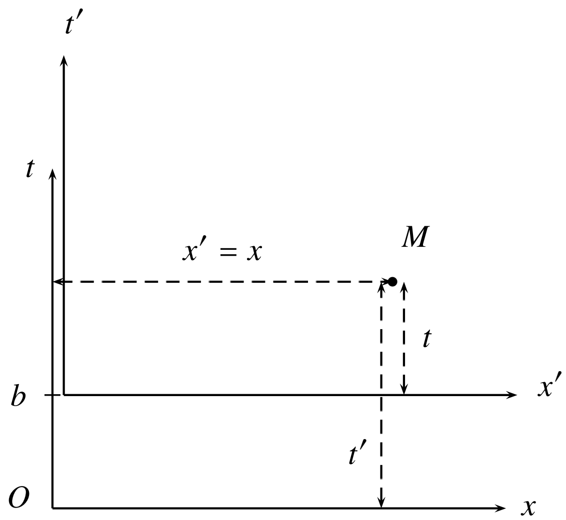
- Translation of the origin of spaces

$$t' = t$$
$$x' = x - a$$



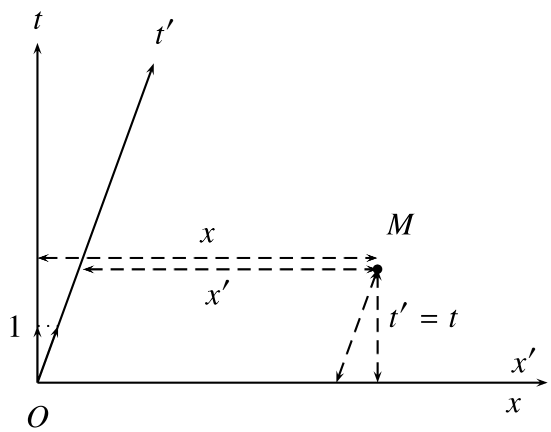
- Translation of the origin of instants

$$t' = t - b$$
$$x' = x$$



- Galilean transformation The abscissa axis is unchanged. The t' axis, set of events occur-

$$t' = t$$
$$x' = x - Vt$$



ring at $x' = 0$ (thus at the origin of $\mathscr{R}'$) is such that

$$x = Vt$$

It is therefore a line passing through the origin and with slope $\frac{1}{V}$ in the old axis system[1].

> One must forget the inevitably Euclidean character of the support (a *spatial* plane in two dimensions) on which this affine diagram is drawn (representing a *space-time* in two dimensions)

So:

- no significance is to be attached to the fact that the (x, t) axes have been drawn orthogonal
- moreover, it is clear that the relation $t' = t$ is not represented by equal lengths. The units on the t and t' axes are not obtained by reporting the same length on both axes.

1.2.2.3 Contradiction between the Galilean theory of relativity and Maxwell's equations

In a reference frame where Maxwell's equations are valid, the propagation of light in vacuum occurs isotropically, with a speed $c = 300\,000$ Km/s. If we apply to this particular speed the Galilean law of velocity composition, it appears that this reference frame must be unique; it is given the name ether. It is unlikely that it is linked to the Earth. However, any attempt to measure the variations in the Earth's speed in the ether fails, while this speed varies, due to the Earth's annual motion. We must therefore conclude the non-existence of the ether, in other words Maxwell's equations are valid in any reference frame. But these same equations are not invariant under Galilean transformations. It is therefore necessary to sacrifice, to lift this contradiction, either Maxwell's equations or classical kinematics, that is, the Galilean theory of relativity.

1.2.2.4 Graphical presentation of "Einsteinian" relativity

Einstein does not bring any new principle; the new theory simply says that the concretization of the principle of relativity in the form of the Galilean transformation is incorrect. It is therefore *another theory to illustrate the same principle.*[2]

a) Assumptions

We will first express that *the speed of light must be the same* in the reference frames $\mathscr{R}$ and $\mathscr{R}'$. In other words, the world line of a photon, with equation

$$x = ct$$

in $\mathscr{R}$, must have the same equation

$$x' = ct'$$

in $\mathscr{R}'$. We will retain the fact that the reference frames $\mathscr{R}$ and $\mathscr{R}'$ are, relative to each other, *in uniform motion* at speed V: that is, in particular that the t' axis (place of events occurring at the origin of $\mathscr{R}'$) has as equation, in the old axis system

$$x = Vt$$

In other words, the t' axis will be the same as in the Galilean theory. For the equation of the photon's world line to be unchanged by passing from $\mathscr{R}$ to $\mathscr{R}'$, it will obviously be necessary to modify the x' axis compared to the Galilean theory; there is one and only one satisfactory choice for this axis.

[1] Attention: *this is an affine diagram*, that is, coordinates are obtained by projections parallel to the axes. The (x, t) plane is obviously not Euclidean since *the notion of right angle has no meaning* given the heterogeneous natures of the quantities on the axes. *There are no distances either*.

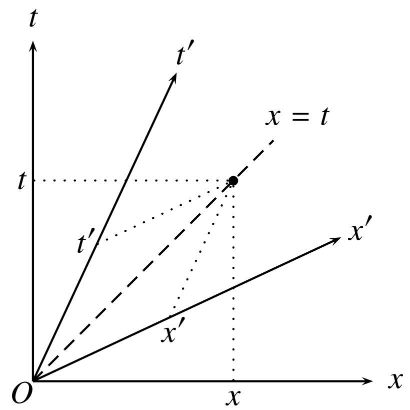
[2] This "presentation" is not a demonstration.

b) Questions of units and constants

To simplify the diagrams and formulas, we agree to choose a system of units such that the speed of light is the unit of speed (for example the unit of time can be the year; then the unit of length will be the light-year).[3] Of course, one can at any time switch back to a traditional unit system (meter and second) by reintroducing c where necessary, with a bit of dimensional analysis.

c) The change of reference frame

With the choice of units such that $c = 1$, the photon's world line will be, in $\mathscr{R}$, the bisector of the two axes (bisector defined as the locus of points such that their affine distances to the two axes are the same). The t' axis has equation $x = Vt$ in the old system.



For the photon's world line to admit the equation $x' = t'$ it is necessary that it also be the bisector of the two new axes. We deduce that the x' axis is the symmetric of the t' axis with respect to the bisector of the first axis system. The equation of the x' axis in the old system is therefore $t - Vx = 0$ We recover the Galilean limit, after making this equation homogeneous:

$$t - \frac{Vx}{c^2} = 0$$

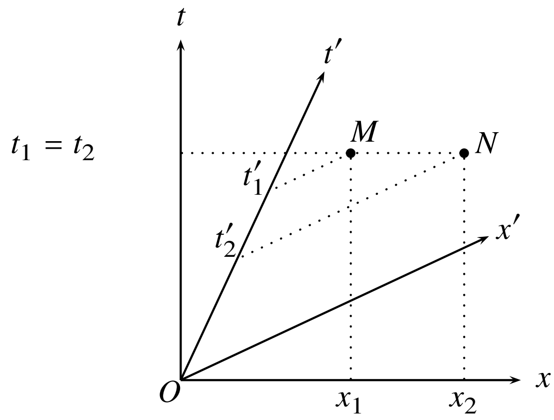
The speed c is then a very large number (which is sometimes abusively expressed by making c "tend" to infinity and the x' axis has almost the equation $t = 0$ It tends to coincide with the x axis.

d) Relativity of simultaneity

This is the first major consequence of Einstein's theory of relativity. Let two events M and N be simultaneous in $\mathscr{R}$; they occur at x_1 and x_2 at the same instant $t_1 = t_2$. The graph immediately shows that $t'_1 \neq t'_2$, unlike the Galilean case where the x and x' axes were coincident.

In the Galilean case, the notion of simultaneity of two events is absolute, that is, independent of the reference frame, in the Einsteinian case, it is relative.

[3] Remark: One must not make the "fundamental" constants of modern physics $(h, c \dots)$ play a role they do not have: they cease to appear in calculations as soon as they are taken as units. Thus, in high-energy physics where all speeds are close to c, it is advantageous to work with $c = 1$; similarly in quantum mechanics where all quantities with the same dimensions as h are precisely of the order of h, it is convenient to take h as unit. Moreover, classical physics, or more simply geometry, show many examples of such an approach (see the adaptation of area units to length units taking into account the law relating the side and area of a square).



e) Calibration of the axes of $\mathcal{R}$ and $\mathcal{R}'$

We have seen that the plane on which the diagram is drawn is not a Euclidean plane; to obtain the unit on the t' axis, we cannot therefore report with a compass the unit chosen on the t axis. We will use *the invariance* (*not demonstrated here*) of the quantity

$$s^2 = t^2 - x^2$$

by change of reference frame. Let us simply note that this invariance is *compatible* with a number of already acquired points:

- compatible with the invariance of c.

  For the photon with equation $x = t$ in $\mathcal{R}$ and $x' = t'$ in $\mathcal{R}'$ we have

$$\left.\begin{aligned} s^2 &= t^2 - x^2 = 0 \\ s'^2 &= t'^2 - x'^2 = 0 \end{aligned}\right\} \quad \text{and so} \quad s^2 = s'^2 = 0$$

- compatible with the Galilean limit.

  By returning to homogeneous notations:

$$s^2 = t^2 - \frac{x^2}{c^2}$$

  if c is very large we have approximately

$$s^2 \approx t^2$$

  Similarly,

$$s'^2 \approx t'^2$$

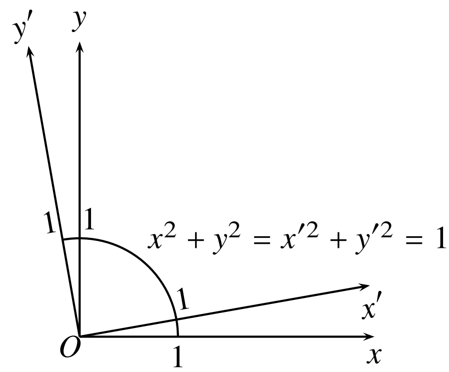
  The invariance of s^2 expresses that $t = t'$ in the Galilean case.

- finally note that in Euclidean geometry the invariant is

$$x^2 + y^2 = x'^2 + y'^2$$

  Here space-time is said to be pseudo-Euclidean, in other words there exists a quadratic norm not positive definite.

Let us then calibrate the axes. In Euclidean geometry the unit on the x' axis would be deduced from the unit on the x or t axis using the fact that the compass traces precisely



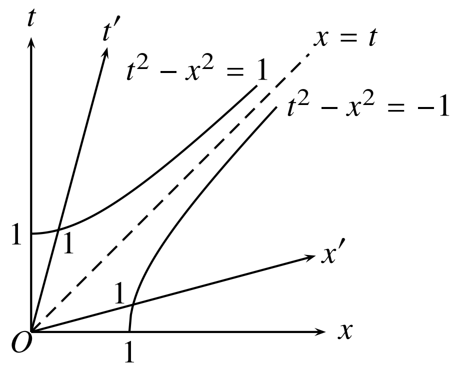
the set of points such that $x^2 + y^2 = x'^2 + y'^2 = 1$. Here, the unit on the t' axis (point with coordinates $t' = 1$ and $x' = 0$) is such that

$$t'^2 - x'^2 = 1 = t^2 - x^2$$

it will therefore be found at the intersection of the t' axis and the hyperbola with equation

$$t^2 - x^2 = 1$$

which plays the role of circles in Euclidean geometry. Similarly the unit on the x' axis

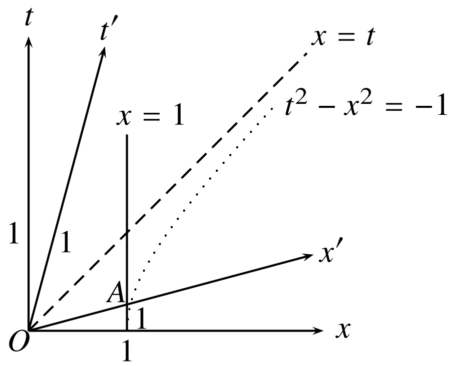


defined by $x' = 1$ and $t' = 0$ is at the intersection of another hyperbola, with equation $t^2 - x^2 = -1$, with the x' axis. We see that these hyperbolas do not cut the line $x = t$; there are therefore no units on this line which cannot constitute either a space axis or a time axis. We therefore do not reach the speed c when V increases. For a photon, there are neither lengths nor times.

f) Contraction of lengths

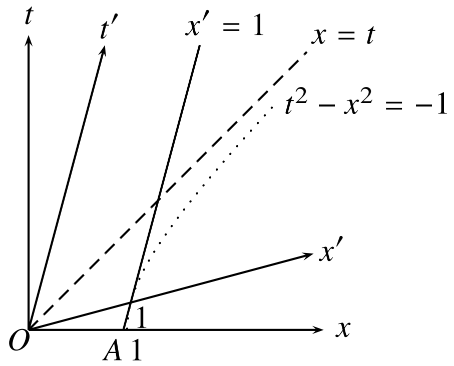
This is only a consequence of the non-absolute character of simultaneity. First define the "measurement of a ruler" process. In a given reference frame the operation consists in determining the abscissas of the two ends of the ruler, at a given instant *of this reference frame*.

So let us therefore consider a unit ruler at rest in $\mathscr{R}$; the world lines of its origin and its end are consequently the two straight lines $x = 0$ and $x = 1$. To measure it in $\mathscr{R}'$, one must choose a definite instant t', for example $t' = 0$. The origin and the end of the ruler are then represented by the points $O(x' = 0)$ and A on the x' axis. Given the position of the hyperbola used for calibrating the axes, the length OA is clearly less than 1. Hence the "phenomenon" of length contraction.



Of course, this contraction is only a perspective effect related to the change of point of view, that is, of reference frame. The ruler does not actually shorten! The only true measurement, without "parallax", is the one performed in the reference frame where the ruler is at rest.

Similarly, a ruler at rest in $\mathscr{R}'$ and measured in $\mathscr{R}$ would give rise to the same contraction effect; the roles played by the two reference frames are symmetric.



g) Time dilation

Muons are unstable particles created by cosmic radiation in the upper atmosphere. Their lifetime τ_0 is on the order of 10^{-6} s. These particles can be observed in high-altitude laboratories (2 or 3 000 m) and their speed V is very close to that of light.

Given that $\tau_0 = 10^{-6}$ s, these particles should on average travel a distance

$$\begin{aligned}\ell &= V\tau_0\\ &= 10^{-6} \times 310^{8}\\ &= 300 \text{ m}\end{aligned}$$

between the place where they are created and the place where they decay. The number of them that travel several kilometers is abnormally high compared to this estimate. This is explained as follows: let $\mathscr{R}$ be the laboratory reference frame, x and t the space-time coordinates of the muon in $\mathscr{R}$. This muon arriving at a speed $V \neq c$ almost constant, its world line is therefore very close to the light line $x = t$.

The event "creation of the muon" being represented at O, the event "disappearance of the muon" is represented at A, on its world line, such that these two events are separated by a proper time τ_0 *in the muon's rest frame*; indeed τ_0 is a proper time, that is, measured in a reference frame in which the two events O and A are seen at the same location, namely the muon's rest frame $\mathscr{R}'$[4].

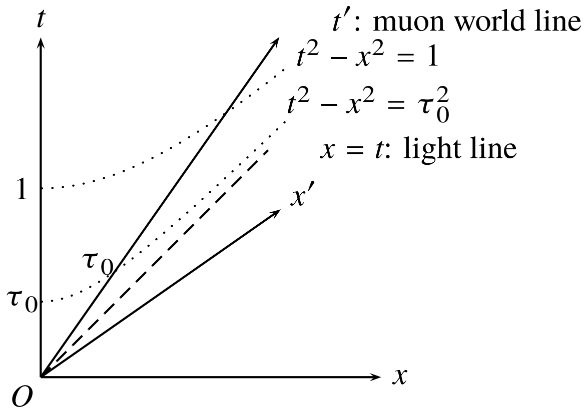


Fig. 1.1: Lifetime of a muon

Now $\mathscr{R}'$ is such that the muon's world line constitutes the t' axis (the x' axis is deduced by symmetry). For observers in $\mathscr{R}$, the instant of A is τ and we have $\tau > \tau_0$ (see figure 1.1). Laboratory observers must therefore perceive a lifetime τ greater than $\tau_0 = 10^{-6}$ s. For them, the average distance traveled before decay will be $V\tau$ and not $V\tau_0$. This is the phenomenon of time dilation, this dilation being of course only apparent because the reference frame $\mathscr{R}$ is not the correct frame for characterizing the history of the muon.

The usual demonstrations of the Lorentz transformation rely on the constancy of the speed of light; this is in particular what was used in the previous graphical presentation. But:

1) This amounts to establishing Einsteinian relativity on a specific property of electromagnetism; doubt is therefore cast on the content of the theory because it seems there is a privileged link between electromagnetism and the structure of space-time; this is what happens in Einstein's paper where the relativity of simultaneity is introduced from the exchange of luminous "signals"; one is then entitled to wonder what other "types of signals" would give.

   Yet all known physical phenomena (gravitation, electromagnetism, strong and weak nuclear interactions) are governed by special relativity; *space-time seems to have a certain structure, independent of the phenomena that occur in it and this is considerably obscured by traditional presentations*.

2) Moreover, the constancy of the speed of light may not be true! The theory of relativity shows that there can exist, alongside particles endowed with mass, other particles, of zero mass, always traveling at the same speed c whatever their energy. It is therefore usually said: "it is because the photon has zero mass that its speed is c". It would be better to say: "If the photon has zero mass, then its speed is c". Since this mass is, at present, surely less than 10^{-60} g, its speed approaches c very closely, enough that it cannot be distinguished from c. But nothing rules out that one day we may detect a non-zero mass for photons and slow them down…

   Then, *the quantity c*, usually called "speed of light", *rather appears as a theoretical*

[4]Note: the lifetime of a particle is always given in its proper frame; only under this condition does it characterize the particle. In practice, however, it is always τ that is measured and τ_0 is deduced by applying the "backward" correction.

structural constant which only measures the speed of the photon if the latter indeed has zero mass (something impossible to prove experimentally). In summary, the constancy of the speed of light therefore constitutes an extremely fragile foundation for Einsteinian relativity. However, this does not mean that the theory itself is fragile: we will see later that it can be based on principles far more solid than the zero mass of the photon. Similarly, Maxwell's electromagnetic theory was originally developed from an ether, an elastic medium seat of a field of mechanical stresses.

Ultimately, this mechanics was abandoned because it was not consistent with the fact that waves propagate in vacuum; Maxwell's equations are nonetheless exact and today they are given other foundations!

3) Einstein's theory of relativity is very likely, just like Galilean relativity, to be called into question one day. If that is the case, *it is important to know exactly what we will actually have to give up in our conception of the physical world* when the theory ceases to be satisfactory. It is therefore appropriate to seek the deepest foundations of the theory instead of continuing to base it on the constancy of the speed of light.

Chapter 2

CONSTRUCTION OF THE LORENTZ TRANSFORMATION FORMULAS

(written by Madeleine Sonneville)

2.1 The Lorentz transformations depend on only one parameter

The principle of relativity therefore asserts that there exist reference frames equivalent for the formulation of the laws of physics. Let $\mathscr{R}(x,t)$ be one of them and $\mathscr{R}'(x',t')$ another. The change-of-frame formulas will therefore be of the type

$$\begin{cases} x' = F(x,t,a_1,a_2,\dots,a_N) \\ t' = G(x,t,a_1,a_2,\dots,a_N) \end{cases}$$

where F and G are two functions to be determined, of the old space-time coordinates x and t, and of a certain number of parameters $\{a_i\}$ that specify the transformation by which $\mathscr{R}$ and $\mathscr{R}'$ correspond. A priori we know neither the number N, nor the physical meaning of these parameters.

2.1.1 Zero hypothesis and choice of origins

Let us make *the hypothesis* that space and time are invariant under translation, in other words that the origins of time and space are arbitrary for the expression of physical laws. This amounts to saying that translations in space

$$x' = x + a \qquad \text{and} \qquad t' = t$$

and in time

$$x' = x \qquad \text{and} \qquad t' = t + b$$

are part of the transformations considered. This hypothesis can obviously be discussed: it constitutes only a local approximation, particularly as regards time, probably valid over durations on the order of a million years; it is no longer true on the scale of the history of an expanding universe.

Assuming this hypothesis, we therefore have a number N of parameters at least equal to 2 (a and b). Let us then focus on transformations other than space-time translations, taking care to "make the origins coincide", that is, to ensure that an event located in $\mathscr{R}$ by $(x = 0,\ t = 0)$, is located in $\mathscr{R}'$ by $(x' = 0,\ t' = 0)$. Then, these "remaining" transformations will no longer contain the parameters a and b and the functions F and G will depend only on $n = N - 2$ parameters. We will therefore have the equations Ⓐ:

$$\text{Ⓐ} \quad \begin{cases} x' = f(x, t, a_1, a_2, \dots, a_n) \\ t' = g(x, t, a_1, a_2, \dots, a_n) \end{cases}$$

such that $x' = 0$ and $t' = 0$ for $x = 0$ and $t = 0$.

2.1.2 N is at most equal to 1

Two arguments support this fact.

2.1.2.1 The existence of causal relations between two events

Let there be an origin event $O(x = 0,\ t = 0\ ;\ x' = a,\ t' = 0)$ and an event E located by x and t in $\mathscr{R}$. Is there a transformation, that is, a set $(a_1, a_2, \dots, a_n)$ such that this event E admits arbitrarily chosen coordinates (x', t') in $\mathscr{R}'$?

The equations Ⓐ must then be solved with respect to the a_i, for given x, t, x', t'. It is clear that if $n > 2$ there will generally be one or more reference frames $\mathscr{R}'$ answering the question. In other words, if for example $t = 1$ hour, it will be possible to find $\mathscr{R}'$ such that $t' = 2$ hours, then to find $\mathscr{R}''$ such that $t = -1$ hour. Any causality between the events O and E is therefore excluded by this fact. Yet these events are arbitrary! We therefore have $n \leqslant 1$, and in particular $n = 0$ if there are no other transformations than space-time translations.

2.1.2.2 The Poincaré principle

It expresses the experimental fact according to which the trajectory of a material point in a reference frame is known as soon as the initial position and initial velocity are determined, while the acceleration is not arbitrary. A change of reference frame is then interpreted as a modification of the initial conditions.

One can therefore always find a reference frame $\mathscr{R}'$ such that the initial conditions (position and velocity) have an arbitrarily chosen value, but the acceleration cannot be arbitrarily chosen, nor can higher-order derivatives. In other words, if we write

$$\begin{cases} x' = f(x, t, a_1, a_2, \dots, a_n) \\ \frac{\mathrm{d}x'}{\mathrm{d}t'} = f^{(1)}\left(x, t, \frac{dx}{dt}, a_1, a_2, \dots, a_n\right) \\ \frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} = f^{(2)}\left(x, t, \frac{dx}{dt}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, a_1, a_2, \dots, a_n\right) \\ \vdots \end{cases}$$

and if this system is considered with respect to the unknowns a_i, we will find the a_i for given x, t and $\mathrm{d}x/\mathrm{d}t$, and for arbitrary x' and arbitrary $\mathrm{d}x'/\mathrm{d}t'$. This would give $n = 2$, but taking into account the choice of origins made above, this number is restricted to $n = 1$ (if $x = 0$ and $t = 0$ we necessarily have $x' = 0$ which is no longer arbitrary)

2.1.3 The nature of space-time and the form of the transformation formulas

So let “a” be the unique parameter defining the transformation $\mathscr{R} \to \mathscr{R}'$. Its physical meaning is unknown for the moment.

2.1.4 Hypothesis 1: homogeneity of space-time

This is a reuse of the zero hypothesis in another form. Let there be, in (x, t) in $\mathscr{R}$ an infinitesimal interval $(\mathrm{d}x, \mathrm{d}t)$. The components of this interval in $\mathscr{R}'$ will be

$$\begin{cases} \mathrm{d}x' &= \dfrac{\partial f}{\partial x}\,\mathrm{d}x + \dfrac{\partial f}{\partial t}\,\mathrm{d}t \\ \mathrm{d}t' &= \dfrac{\partial g}{\partial x}\,\mathrm{d}x + \dfrac{\partial g}{\partial t}\,\mathrm{d}t \end{cases}$$

If space-time is homogeneous, $\mathrm{d}x'$ and $\mathrm{d}t'$ depend neither on x nor on t and therefore $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial t}$ depend neither on x, nor on t, and are therefore constants. *The functions f and g are linear in x and t*, that is:

$$\text{Ⓑ} \quad \begin{cases} x' = H(a)x - K(a)t \\ t' = L(a)t - M(a)x \end{cases}$$

Homogeneity results from the zero hypothesis on the choice of origins.
Consequence: a point attached to $\mathscr{R}'$, that is, such that $x' = c^{ste}$, will satisfy

$$x = \frac{K(a)}{H(a)}\,t + \frac{x'}{H(a)}$$

Its motion is uniform. We now know that *the equivalent reference frames $\mathscr{R}$ and $\mathscr{R}'$ are in uniform motion relative to each other.* This is not a hypothesis but follows from the homogeneity of space-time.

Change of parameter: We see that $\frac{K(a)}{H(a)}$ has the dimensions of a velocity; let V be this quantity that we make appear in x' and t':

$$\begin{cases} x' &= H(a)[x - Vt] \\ t' &= H(a)\left[\dfrac{L(a)}{H(a)}\,t - \dfrac{M(a)}{H(a)}\,x\right] \end{cases}$$

Let us set

$$H(a) = \gamma(V) \qquad \frac{L(a)}{H(a)} = \lambda(V) \qquad \frac{M(a)}{H(a)} = \mu(V)$$

We will have the equations:

$$\text{Ⓒ} \quad \begin{cases} x' = \gamma(V)[x - Vt] \\ t' = \gamma(V)[\lambda(V)t - \mu(V)x] \end{cases}$$

where only three functions remain to be determined, of the parameter V, instead of the four functions of a present in equations Ⓑ.

Counter-example 1: model not satisfying hypothesis 1

$$\begin{cases} x' = x - V\tau \sinh\left(\dfrac{t}{\tau}\right) \\ t' = t \end{cases}$$

The transformation is not linear; τ is a constant of the Universe (“age”). This model admits a Galilean limit if $t \ll \tau$:

$$\begin{cases} x' \approx x - Vt \\ t' = t \end{cases}$$

2.1.5 Hypothesis 2: isotropy of space

In three dimensions, this is invariance under rotation; here, it will be limited to invariance under reversal of the orientation of the space axis. Let there be two equivalent reference frames $\mathscr{R}$ and $\mathscr{R}'$ characterized by V; x and t being the coordinates of an event E in $\mathscr{R}$, its coordinates in $\mathscr{R}'$ are obtained by equations Ⓒ. Let us call $\bar{\mathscr{R}}$ and $\bar{\mathscr{R}}'$ the reference frames obtained by reversing the orientation of the abscissa axis of $\mathscr{R}$ and $\mathscr{R}'$. The isotropy of space requires that $\bar{\mathscr{R}}$ and $\bar{\mathscr{R}}'$ are also equivalent, left or right has no intrinsic meaning, the orientation of the axes is arbitrary. Therefore one passes from $(-x, t)$ to $(-x', t')$ by formulas of type Ⓒ, the parameter no longer being V but U, to be determined as a function of V, that is, for all t and for all x:

$$\begin{cases} -x' = \gamma(U)[-x - Ut] \\ t' = \gamma(U)[\lambda(U)t + \mu(U)x] \end{cases}$$

By identifying the two expressions of x' and t':

$$\begin{cases} \gamma(V) = \gamma(U) \\ V\gamma(V) = -U \cdot \gamma(U) \\ \gamma(V)\lambda(V) = \gamma(U)\lambda(U) \\ \gamma(V)\mu(V) = -\gamma(U)\mu(U) \end{cases}$$

Therefore

$$\boxed{V = -U}$$

which is consistent with intuition. By reversing the direction of the axis in $\mathscr{R}$, we change the sign of the velocity of the other reference frame in $\mathscr{R}$.

γ is an even function of V

λ is even

μ is odd

Counter-examples 2: models not satisfying hypothesis 2

$$\begin{cases} x' = e^{\sigma V}(x - Vt) \\ t' = e^{\sigma V}t \end{cases}$$

where $e^{\sigma V}$ plays the role of γ but is not even.
Similarly for

$$\begin{cases} x' = \dfrac{x - Vt}{1 + \rho V} \\ t' = \dfrac{t - \rho^2 Vx}{1 + \rho V} \end{cases}$$

where the term $(1 + \rho V)$ is not an even function of V.

2.1.6 Hypothesis 3: the set of Lorentz transformations constitutes a group

2.1.6.1 This hypothesis is by far the strongest

Let us first note that *physical equivalence* between two reference frames $\mathscr{R}$ and $\mathscr{R}'$ is naturally an *equivalence relation*, in the mathematical sense of the term

- $\mathscr{R}$ is equivalent to $\mathscr{R}$(reflexivity)
- if $\mathscr{R}$ is physically equivalent to $\mathscr{R}'$, then $\mathscr{R}'$ is physically equivalent to $\mathscr{R}$(symmetry)
- if $\mathscr{R}$ and $\mathscr{R}'$ are physically equivalent and the same holds for $\mathscr{R}'$ and $\mathscr{R}''$, then $\mathscr{R}$ and $\mathscr{R}''$ are physically equivalent (transitivity)

2.1.6.2 Meaning of this hypothesis

Let T be a Lorentz transformation, that is, a transformation by which two physically equivalent reference frames correspond, and let $(\mathscr{T})$ be the set of Lorentz transformations, equipped with the "composition of transformations" law denoted $*$. Then this law must

1) be internal, that is,

$$\forall T_1 \in (\mathscr{T}) \;:\quad T_2 * T_1 = T_3 \quad \text{and} \quad T_3 \in (\mathscr{T})\ \forall T_2 \in (\mathscr{T})$$

   In other words if T_1 allows passage from $\mathscr{R}$ to $\mathscr{R}'$ and T_2 from $\mathscr{R}'$ to $\mathscr{R}''$, then one passes from $\mathscr{R}$ to $\mathscr{R}''$ by a Lorentz transformation. This therefore translates *transitivity*

2) possess a neutral element, that is,

$$\exists T_0,\ T_0 \in (\mathscr{T}) \qquad \text{such that} \qquad \forall T \in (\mathscr{T})\ : T * T_0 = T$$

   T_0 is an identity transformation that makes a reference frame correspond to itself; and T_0 is part of the Lorentz transformations. The existence of T_0 translates reflexivity

3) every transformation T must possess an inverse, denoted T^{-1}

$$\forall T \in (\mathscr{T}),\ \exists T^{-1} \in (\mathscr{T}) \qquad \text{with} \qquad T^{-1} * T = T_0$$

   If T transforms $\mathscr{R}$ into $\mathscr{R}'$, we go back from $\mathscr{R}'$ to $\mathscr{R}$ by T^{-1} which is also a Lorentz transformation. This illustrates symmetry

4) finally the associative nature of the composition law is automatic since it is a set of transformations (and not just any set)

> The requirement of the group law therefore only mathematically translates the principle of relativity, that is, the existence of equivalent reference frames.

Counter-examples 3:

$$\begin{cases} x' = x - Vt \\ t' = t - \dfrac{Vx}{c^2} \end{cases}$$

This set of transformations does not have a group structure: the transformation T_0 exists but the law is not internal.

The first-order expansion in $\frac{1}{c^2}$ of the Lorentz transformation (see p. 26).

$$\begin{cases} x' = (x - Vt)\left(1 + \dfrac{V^2}{2c^2}\right) \\ t' = t\left(1 + \dfrac{V^2}{2c^2}\right) - \dfrac{Vx}{c^2} \end{cases}$$

does not correspond to a group structure.

2.1.6.3 What does the group law imply?

a) the identity transformation is the Lorentz transformation T_0 such that $x' = x$ and $t' = t$. It therefore obviously corresponds to $V = 0$. Since it is part of $(\mathscr{T})$, we have:

$$\boxed{\gamma(0) = 1 \qquad \lambda(0) = 1 \qquad \mu(0) = 0}$$

b) the inverse transformation is a Lorentz transformation. Now, by inverting equations Ⓒ we find:

$$\begin{cases} x = \dfrac{1}{\gamma(V)}\left[x' + Vt'\right] \\ t = \dfrac{1}{\lambda(V)}\left[t' + \mu(V)x'\right] \end{cases}$$

There must therefore exist a parameter W characterizing the transformation $\mathscr{R}' \to \mathscr{R}$, such that this can be written

$$\begin{cases} x = \gamma(W)\,(x' - Wt') \\ t = \gamma(W)\,[\lambda(W)t' - \mu(W)x'] \end{cases}$$

from which the conditions on *identity of the terms in* t'

$$W = -\frac{V}{\lambda(V)} \qquad \text{and} \qquad \lambda(W) = \frac{1}{\lambda(V)}$$

from which we deduce a functional equation in λ:

$$\lambda\left[-\frac{V}{\lambda(V)}\right]^{-1} = \frac{1}{\lambda(V)}$$

Note that it suffices to determine a function λ for positive arguments V to deduce the function for negative arguments. Let us perform the change of variable defined by:

$$Z(V) = \frac{V}{\lambda(V)}$$

then the functional equation is written:

$$Z[-Z(V)] = -V \qquad \text{or} \qquad Z(V) = -Z^{-1}(-V)$$

The condition $Z[-Z(V)] = -V$ can only be satisfied if the graphical representation of $Z(V)$ *is symmetric with respect to the second bisector*[1].

Moreover, since $\lambda(V)$ is even, $Z(V)$ is odd; therefore the graphical representation is symmetric with respect to O and therefore also with respect to the first bisector.

We also know that $\lambda(0) = 1$. The only possibility is therefore $Z(V) = V$ so

$$\lambda(V) = 1$$

We deduce

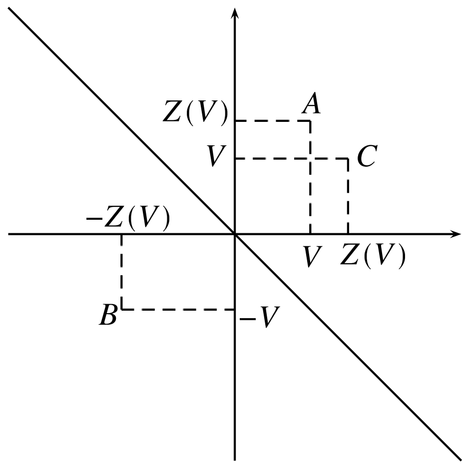
$$W = -V$$

The parameter of the transformation $\mathscr{R}' \to \mathscr{R}$ is therefore $-V$.

Let us then exploit *the identity of the terms in* x':

$$\begin{aligned} \gamma(W) &= \gamma(-V) \\ &= \gamma(V) \\ &= \frac{1}{\gamma(V)}[1 - V\mu(V)]^{-1} \end{aligned}$$

[1]Indeed if A, with coordinates V and $Z(V)$ is a point on the curve, the point B with abscissa $-Z(V)$ and ordinate $-V$, symmetric to A with respect to the second bisector, must be on the graphical representation of $Z(V)$ since $Z(-Z(V)) = -V$



$$\boxed{[\gamma(V)]^2[1 - V\mu(V)] = 1}$$

The condition $-\mu(W) = \frac{\mu(V)}{\lambda(V)} = \mu(V)$ is automatically satisfied since $V = -W$ and μ is odd.

c) The composition law is internal. Let us make two Lorentz transformations of parameters V_1 and V_2 follow one another. We will have:

$$\begin{cases} x' = \gamma(V_1)\,(x - V_1 t) \\ t' = \gamma(V_1)\,[t - \mu(V_1)x] \end{cases} \qquad \begin{cases} x'' = \gamma(V_2)\,(x' - V_2 t') \\ t'' = \gamma(V_2)\,[t' - \mu(V_2)x'] \end{cases}$$

that is:

$$\begin{cases} x'' = \gamma(V_1)\gamma(V_2)\,[1 + \mu(V_1)V_2]\left[x - \dfrac{V_1 + V_2}{1 + V_2\mu(V_1)}t\right] \\ t'' = \gamma(V_1)\gamma(V_2)\left[t - \dfrac{V_1 + V_2}{1 + V_1\gamma(V_2)}x\right] \end{cases}$$

If this is a Lorentz transformation, we must find $\bar{V}$ such that:

$$\begin{cases} x'' = \gamma(\bar{V})\left(x - \bar{V}t\right) \\ t'' = \gamma(\bar{V})\left[t - \mu(\bar{V})x\right] \end{cases}$$

And therefore

$$\begin{aligned} \gamma(\bar{V}) &= \gamma(V_1)\gamma(V_2)\,[1 + \mu(V_1)V_2] \\ &= \gamma(V_1)\gamma(V_2) \end{aligned}$$

$$[1 + V_1\mu V_2]\,\mu(V_1)V_2 = \mu(V_2)\cdot V_1$$

so

$$\frac{\mu(V_1)}{V_1} = \frac{\mu(V_2)}{V_2}$$

Since the arguments V_1 and V_2 are arbitrary, we must have

$$\frac{\mu(V)}{V} = \alpha$$

with α constant. Therefore

$$\gamma(V) = +\frac{1}{\sqrt{1 - \alpha V^2}}$$

positive sign because $\gamma(0) = 1$. And finally

$$\begin{cases} x' = \dfrac{x - Vt}{\sqrt{1 - \alpha V^2}} \\ t' = \dfrac{t - \alpha V x}{\sqrt{1 - \alpha V^2}} \end{cases}$$

Moreover, we obtain

$$\bar{V} = \frac{V_1 + V_2}{1 + \alpha V_1 V_2}$$

The number α obviously depends on the unit system. Its numerical value is of no interest, only its sign or possible nullity matters.

- case $\alpha = 0$

$$\begin{cases} x' = x - Vt \\ t' = t \end{cases}$$

  This is the Galilean transformation.
- case $\alpha > 0$
  We can then write $\alpha = \frac{1}{c^2}$ hence the classical expression of the Lorentz transformation. But we do not know what "c" is the speed of and nothing guarantees that objects of speed c exist in nature. "c" *is a structural constant of space-time*.
- case $\alpha < 0$
  Let us denote $\alpha = -\frac{1}{K^2}$ where K is a speed. Then:

$$\begin{cases} x' = \dfrac{x - Vt}{\sqrt{1 + \frac{V^2}{K^2}}} \\ t' = \dfrac{t + \frac{Vx}{K^2}}{\sqrt{1 + \frac{V^2}{K^2}}} \end{cases}$$

2.1.7 Hypothesis 4: existence of causality

There exist intervals whose temporal component sign is invariant. In other words pairs of events such that the statement "B occurs after A" is intrinsic. Then,

- if $\alpha = 0$
  Time being invariant, the statement "B occurs after A" is intrinsic for *all* pairs of events
- if $\alpha < 0$
  Let A be the origin event and B the event (x, t). One can always find V such that t' is negative if t is positive. Causality does not exist
- if $\alpha > 0$
  We must have $1 - \frac{V^2}{c^2} > 0$, so $|V|$ is bounded by c

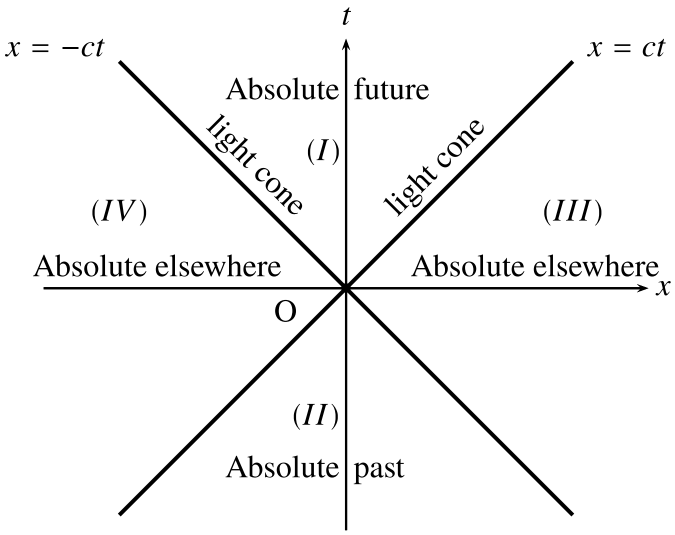
For $t > 0$ the value of $|V|$ necessary to make t' negative is outside the interval $[0, c]$. Causality will exist and this case is therefore suitable. Hence the various types of intervals, space-like and time-like:

- if $\left|\frac{\Delta x}{\Delta t}\right| > c$
  Space-like intervals, no causality, "before" and "after" interchangeable

- if $\left|\frac{\Delta x}{\Delta t}\right| < c$
  Time-like intervals, causality exists

Graphical representation of space-time

Let there be an origin event O and a reference frame $\mathscr{R}$ in which every event E is represented by its coordinates x and t in the plane. Relative to this origin, the set of events is ordered into various classes that region the plane.



- events whose representative points are on the lines $x = \pm c$. They can be causally related to the origin event. They are said to belong to the "light cone", in the sense that the signal connecting them can be electromagnetic in nature (if c indeed represents the speed of light)
- events in regions (I) and (II) are such that $\left|\frac{x}{t}\right| < c$. The interval between the considered event and the origin event is time-like. Causal relation is possible between them. Region (I) constitutes the absolute future of the origin event. Whatever the reference frame, event E will be after the origin event, although the instant t attributed to it depends on the reference frame. Similarly (II) is the absolute past of the origin event.
- Events in region (III) cannot be causally related to the origin event because $\left|\frac{x}{t}\right| > c$. The order in which events O and E succeed each other varies with the observer. One never finds a reference frame such that the events occur at the same place. We are in *absolute elsewhere*. Conversely, in (I) and (II) one can find a reference frame such that the events occur at the same place.

Link with the Galilean case

If c appears as a very large quantity, region (III) degenerates because the light cone flattens onto the x axis. Absolute elsewhere is reduced to the present! And therefore, apart from events simultaneous with the origin event, all other events are either past, or future and this in an absolute way. *We therefore see that Einsteinian space-time differs from Galilean space-time by the fact that there exists a large class of events without causal relation with an origin event. Causality is therefore much stronger in Galilean physics.*

2.1.8 The concept of velocity in Galilean and Einsteinian kinematics

Let us rewrite the Lorentz transformation formulas with $c = 1$:

$$\begin{cases} x' = \dfrac{x - Vt}{\sqrt{1 - V^2}} \\ t' = \dfrac{t - Vx}{\sqrt{1 - V^2}} \end{cases}$$

Let then

$$\mathscr{T} = \frac{1}{\sqrt{1 - V^2}} \qquad \text{and} \qquad \mathscr{S} = \frac{V}{\sqrt{1 - V^2}}$$

We have in particular

$$\mathscr{T}^2 - \mathscr{S}^2 = 1$$

and we can set $\mathscr{T} = \cosh(\varphi)$ and $\mathscr{S} = \sinh(\varphi)$, hence

$$\begin{cases} x' = \cosh(\varphi)x - \sinh(\varphi)t \\ t' = \cosh(\varphi)t - \sinh(\varphi)x \end{cases}$$

So compared to Euclidean space, the transformations are "almost" rotations: they are hyperbolic transformations.

Interest of this writing

We have seen that by composing two Lorentz transformations of parameters V_1 and V_2 we obtain a transformation of parameter

$$\bar{V} = \frac{V_1 + V_2}{1 + V_1 V_2}$$

By setting $V = \frac{\mathscr{S}}{\mathscr{T}} = \tanh(\varphi)$ ($| V |< 1$), we will have:

$$\begin{aligned} \bar{V} &= \tanh(\bar{\varphi}) \\ &= \frac{V_1 + V_2}{1 + V_1 V_2} \\ &= \frac{\tanh(\varphi_1) + \tanh(\varphi_2)}{1 + \tanh(\varphi_1)\tanh(\varphi_2)} \\ &= \tanh(\varphi_1 + \varphi_2) \end{aligned}$$

and therefore

$$\bar{\varphi} = \varphi_1 + \varphi_2$$

> When composing two Lorentz transformations, the parameter φ is additive whereas V is not.

φ is therefore the *right* parameter for labeling a Lorentz transformation. Not using φ for this purpose is as absurd as refusing to use angles to parametrize rotations in traditional geometry, and only using their tangents! φ is called *rapidity*.

This illustrates the following theorem: "every continuous one-parameter group is isomorphic to the additive group of the reals". Among all possible parameters (functions of one another) there exists one that is additive; this is the case of the group of rotations whose additive parameter is *the angle*; for the Lorentz group, this parameter is φ, the rapidity.

2.1.8.1 Galilean velocity and Einsteinian velocity

In the Galilean case, the concept of velocity covers two quite distinct properties.

1) it is first the time rate of change of position $V = \frac{dx}{dt}$
2) when composing two motions, this parameter is additive, $\bar{V} = V_1 + V_2$

It automatically follows that the values of this parameter describe the real line. When moving to Einsteinian kinematics, nothing guarantees that these two properties will continue to be carried by the same concept. Indeed:

1) property 1) is still that of the quantity called velocity $V = \frac{dx}{dt}$
2) but the additive quantity (and which takes any real value) is no longer V but φ

<u>A false paradox</u>

This demystifies the pseudo-paradox according to which

$$c + \text{finite velocity} = c$$

Indeed it is the φ that add up. For $V = c$ we have φ infinite and consequently, by adding the φ we have:

$$\varphi_{infinite} + \varphi_{finite} = \varphi_{infinite}$$

which is no longer paradoxical at all.

In the Galilean limit, moreover, with $V = c\tanh(\varphi)$ and $V \ll c$: $\tanh(\varphi) \ll 1$ therefore $\varphi \ll 1$ and

$$\boxed{\tanh(\varphi) \approx \varphi}$$

There is no longer any distinction between the concepts.

This approach to relativity and the Lorentz relations themselves are only concerned with the space-time coordinates of events and their transformation from one reference frame to another. Yet in "relativity factories" such as laboratories like CERN, equipped with powerful accelerators, all measurements made to characterize particles concern energies and momenta. Another introduction to relativity can be made from these quantities. Our goal here has not been to treat both approaches simultaneously but one will find in the appendix a sketch of the approach followed from a dynamic perspective.

2.1.9 Appendix

2.1.9.1 Dynamic aspects of the theory

(written by Mireille Rousseau)

In Galilean theory momentum and energy are expressed by:

$$\vec{p} = m\vec{v}$$
$$E = \frac{1}{2}mv^2 + U$$

v being the velocity, m the mass, U the internal energy of the particle. Can these concepts be generalized in the framework of Einsteinian theory? It seems so since the existence of radiation pressure proves the presence of energy and inertia in the electromagnetic field.

Let us define inertia $I(v)$ by the relation[2]:

$$\vec{p} = I\vec{v}$$

[2] Inertia varies with velocity and should not be confused with mass m which is something intrinsic; mass m corresponds to rest inertia; it is therefore redundant to speak of "rest mass".

We can also write the following two relations[3]:

$$v = \frac{\mathrm{d}E}{\mathrm{d}p}$$
$$\mathrm{d}I = \alpha \mathrm{d}E$$

From these three basic relations it is possible to deduce p, E, I as functions of v since we have three relations for four unknowns. It suffices to write:

$$\begin{aligned}
\mathrm{d}p &= d(Iv) \\
&= I\mathrm{d}v + v\mathrm{d}I \\
&= \frac{p}{v}\mathrm{d}v + v\alpha\mathrm{d}E \\
&= \frac{p}{v}\mathrm{d}v + v^2\alpha\mathrm{d}p \\
\mathrm{d}p\left(1 - \alpha v^2\right) &= \frac{p}{v}\mathrm{d}v \\
\frac{\mathrm{d}p}{p} &= \frac{\mathrm{d}v}{v\left(1 - \alpha v^2\right)} \\
&= \frac{\mathrm{d}v}{v} + \frac{\alpha v \mathrm{d}v}{1 - \alpha v^2}
\end{aligned}$$

so

$$\begin{aligned}
\ln(p) &= \ln(v) - \tfrac{1}{2}\ln(1 - \alpha v^2) \\
&= \ln\left(\frac{v}{\sqrt{1 - \alpha v^2}}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
p &= C^{ste}\frac{v}{\sqrt{1 - \alpha v^2}} \\
I &= \frac{c^{ste}}{\sqrt{1 - \alpha v^2}} \quad \text{the constant can be denoted } I_0 \\
E &= \frac{I_0}{\alpha\sqrt{1 - \alpha v^2}}
\end{aligned}$$

These three expressions can also be written ($m = C^{ste}$ and $\alpha = c^2$):

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$I = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We can deduce from this the fundamental invariant relation of relativistic dynamics:

$$E^2 - p^2c^2 = m^2c^4$$

[3]Adding energy increases inertia.

or, in a more convenient unit system where $c = 1$

$$\boxed{E^2 - p^2 = m^2}$$

which amounts to saying that the length of the energy-momentum quadrivector is an invariant quantity under change of reference frame, therefore independent of the velocity of the particle considered. It is this relation that allows simple calculations on nuclear reactions, whereas a calculation by Lorentz transformation would quickly become tedious.

Let us take as an example the production of an antiproton by collision of two protons,

$$p + p \to \bar{p} + p + p + p$$

and find what is the minimum energy E necessary for a proton hitting another stationary target proton for there to be creation of an antiproton. We are at the threshold of the reaction if, in the center-of-mass frame, the three protons and the antiproton obtained after interaction are at rest. So, in this frame:

$$\begin{aligned} \sum \vec{p} &= \vec{0} \\ \sum E &= 4m \end{aligned}$$

The square of the length of their energy-momentum quadrivector is therefore $(4m)^2$. Now before the reaction, in the laboratory frame we could calculate the square of the length of the energy-momentum quadrivector for the two basic protons, that is:

$$\begin{aligned} \left(\sum E\right)^2 - \left(\sum p\right)^2 &= (E + m)^2 - p^2 \\ &= 2m(E + m) \end{aligned}$$

quantity invariant under passage to the center-of-mass frame[4].
To finish solving the problem, we only need to note that during the interaction, there was conservation of the total energy of the system and of its total momentum, therefore of the length of the corresponding quadrivector, that is:

$$\begin{aligned} 2m(E + m) &= (4m)^2 \\ E &= 7m \quad \text{that is a kinetic energy } E_c = 6m \end{aligned}$$

We have been led here to distinguish *invariant* quantities[5] (under change of reference frame) and *conserved* quantities[6] (during the interaction), some being able to belong to both categories[7]. It is therefore probably good to draw up a comparative table of invariant and conserved quantities in Galilean and Einsteinian relativities.

[4]We know well the initial state of the system in the laboratory frame, the final state in the center-of-mass frame. It is this passage from one frame to another that would quickly become inextricable by Lorentz transformation.

[5]mass and length of the energy-momentum quadrivector.

[6]total energy, total momentum, length of the associated quadrivector.

[7]here, the length of the energy-momentum quadrivector.

quantity	Galilean	Einsteinian
energy E	non-invariant-conserved	non-invariant-conserved $E = Ic^2$
inertia I	invariant-conserved $I = m$	non-invariant-conserved $I = E/c^2$
mass m	invariant-conserved $m = I$	invariant-not conserved $m = U/c^2$
internal energy U	invariant-not conserved	invariant-not conserved $U = mc^2$

We note in this table that in the Galilean case there exists one more independent quantity than in the Einsteinian case, but one more conservation law. There are therefore as many constraints in one theory as in the other. Moreover it should be noted that inertia is a "pivotal quantity" between the two theories: in one it is identified with mass, in the other with energy.

Chapter 3

RELATIVISTIC ELECTROMAGNETISM

(written by Claude Maître)

3.1 Maxwell's equations

The purpose of this article is to show how Maxwell came up with the idea of adding *one term* to a set of equations that were already implicitly known, and how, only from that moment on, the problem of Einsteinian relativity arose. This term is the one called the "displacement current" (which, incidentally, has nothing to do with a current).

Let us first take stock of what was known until then in electromagnetism. It was known that:

1) the sources of the electric field are charges, which is expressed as:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{3.1}$$

where ρ is the electric charge density.

2) the induction phenomenon, known since 1830, is expressed as:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3) the magnetic field obeys the equation:

$$\vec{\nabla} \cdot \vec{B} = 0$$

This equation expresses the absence of isolated magnetic poles. We will come back to this later.

4) the sources of the magnetic field are electric currents (Ampère, Oersted 1820):

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{3.2}$$

Ampère's hypothesis consists in saying that only these currents are sources, and that there are no magnetic charges. In the case of a magnet, the magnetic field is due to microscopic currents circulating within it; these currents are those of the atomic electrons.

Here we use modern notation (div $\vec{E}$ or $\vec{\nabla} \cdot \vec{E}$) whereas Maxwell still wrote

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

which makes the writing almost unreadable even at the level of notation itself.

We are familiar with the div notation but not with the four-dimensional Minkowski relativistic notations, which are even more condensed, and which perform, with respect to classical writing, the same type of condensation as the latter does with respect to Cartesian coordinate writing. Here we have an example of the evolution of formalism and notation, which is very interesting in the history of physics. We realize that changes in notation conceal conceptual changes that are not always explicit. The transition from $\partial_x E_x + \partial_y E_y + \partial_z E_z$ to $\vec{\nabla} \cdot \vec{E}$ shows that we are truly dealing with vector analysis, that $\vec{E}$ is a vector just like $\vec{\nabla}$, and that together they form a scalar; in other words, it is the entire invariance of the theory under spatial rotations that is brought to light by this notation. This character does not appear directly in Cartesian notation. There is therefore, in this evolution of notation, a real deepening of thought.

The same phenomenon occurs when we move from this notation where spatial and temporal aspects are separated, to a four-dimensional writing that reveals the transformation properties of the electromagnetic field under Lorentz transformations.

From a pedagogical point of view, it is perhaps good to emphasize that notations are not neutral and do not merely concern questions of pure convenience.

These four expressions were therefore known before Maxwell. However, for the fourth equation to be exact, it is necessary that $\vec{\nabla} \cdot \vec{J} = 0$, in other words, that the electric current be conserved. Examples can be given where this condition is not satisfied.

Example: Consider a sphere covered with a radioactive α-emitting material. The emission is assumed to be radial. There is therefore a radial current leaving this sphere. One would therefore expect the existence of a magnetic field. We will try to determine the characteristics of $\overset{\curvearrowleft}{B}$ using spatial symmetry arguments[1].

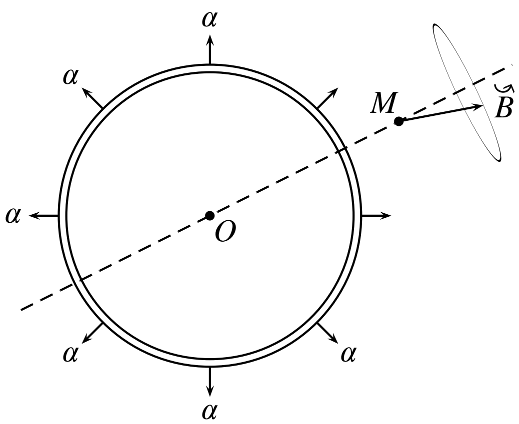


Fig. 3.1: Radioactive sphere

The spatial situation of the example possesses spherical symmetry. The vector field created can therefore only be radial. Suppose that at any point M, $\overset{\curvearrowleft}{B}$ is not radial. During a rotation around a diametral axis passing through M (fig. 3.1), the field $\overset{\curvearrowleft}{B}$ describes a cone, while the source system remains unchanged. This contradicts the principle that the symmetry of the causes must

[1]This type of argument is very important in physics. In essence, the theory of relativity is nothing more than the exploitation of symmetry properties in space-time

be reflected in the symmetry of the effects: therefore $\overset{\curvearrowleft}{B}$ can only be radial and its magnitude can only depend on the distance from M to the center of the sphere.
On the other hand, the magnetic field is a pseudovector, which gives it certain very particular properties with respect to reflection (see appendix). Imagine a mirror cutting the sphere by a diametral plane passing through M (fig. 3.2). $\overset{\curvearrowleft}{B}$ is therefore parallel to the mirror.

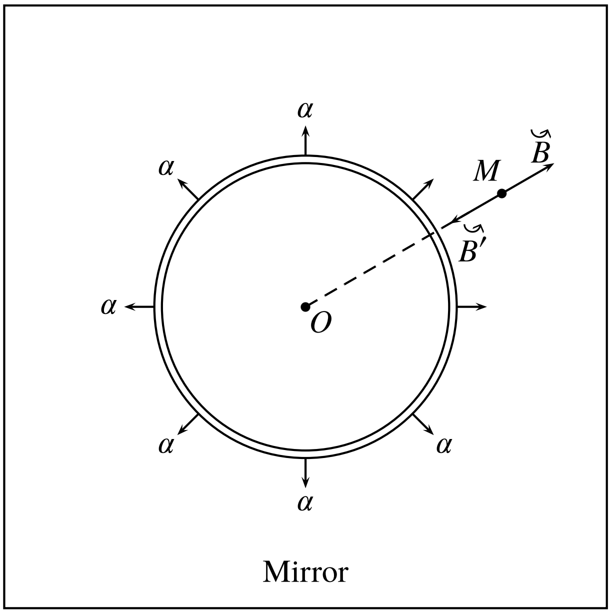


Fig. 3.2: Mirror intersecting the sphere in a diametral plane

The image of the source is still unchanged as a geometric object (sphere) but also as a physical source (electric charges are invariant by reflection). But $\overset{\curvearrowleft}{B}$, parallel to the mirror, is reversed by reflection. We find the same paradox as before, so $\overset{\curvearrowleft}{B}$ is zero which is incompatible with $\vec{\nabla} \times B = \mu_0 \vec{J}$ because $\vec{\nabla} \cdot \vec{J}$ is not zero. Equation (3.2) is therefore incorrect in this case.
Here we must ensure the total conservation of charge, although the charge of the sphere decreases. The charge conservation equation is:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}$$

where I is the outgoing current and q the internal charge. This remains valid for any volume. Let $\vec{J}$ be the current density vector:

$$\begin{aligned} I &= \text{flux of } \vec{J} \\ &= \iiint \vec{\nabla} \cdot \vec{J}, \mathrm{d}v \\ &= -\frac{d}{dt} \iiint \rho, \mathrm{d}v \end{aligned}$$

$$\Longrightarrow \qquad \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

One may wonder what relation exists between this conservation equation and relativity. The local conservation of charge is not imposed by Galilean theory. One can imagine a mechanism perfectly compatible with a Galilean space-time, in which charge is globally conserved in the universe, but not locally: a charged particle disappears at one point, for example it disintegrates into uncharged particles; if, at the same moment, elsewhere in space a charged particle is created, the total charge is conserved. The difficulty comes from the simultaneity of events, possible in a Galilean space-time. But space-time is Einsteinian and simultaneity is only possible in one reference frame and not in others, unless however the two phenomena (appearance and disappearance of charge) occur at the same place. Conservation must therefore be local. This confirms the need to replace $\vec{\nabla} \cdot \vec{J} = 0$ (imposed by equation (3.2) in its pre-Maxwellian form) by:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

This expression can moreover be written in Einsteinian notation using the charge and current density quadrivector:

$$\underline{\partial} \cdot \underline{J} = 0 \qquad \begin{cases} \underline{\partial} = \left(-\frac{\partial}{\partial t}, \vec{\nabla} \right) & \text{quadriderivation vector} \\ \underline{J} = (\rho, \vec{J}) & \end{cases}$$

$\underline{\partial} \cdot \underline{J}$ is a pseudo-scalar product in a four-dimensional space.
ρ, temporal component of $\underline{J}$, transforms like t.
$\vec{J}$, spatial component of $\underline{J}$, transforms like $\vec{x}$.
This writing shows that ρ and $\vec{J}$ transform like t and $\vec{x}$ in a Lorentz transformation. In the same way, all Maxwell's equations can be written in four-dimensional form, which explicit their transformation properties by Lorentz transformations. In summary, if Einsteinian relativity is true, there is *local* conservation of charge. If there is local conservation of charge, then the equation

$$\vec{\nabla} \cdot \overset{\curvearrowleft}{B} = \mu_0 \vec{J}$$

is false because it imposes $\vec{\nabla} \cdot \vec{J} = 0$, which is not always verified. It is therefore necessary to modify at least equation (3.2), preserving linearity in $\vec{E}$ and $\overset{\curvearrowleft}{B}$ (which leads to the superposition principle) and the first order of derivation (which allows limiting initial data to those of fields on a surface). We therefore seek a term $\vec{X}$ satisfying these conditions:

$$\begin{aligned} \vec{\nabla} \cdot \overset{\curvearrowleft}{B} = \mu_0 \vec{J} + \vec{X} \quad &\Longrightarrow \quad \vec{\nabla} \cdot (\vec{\nabla} \times \overset{\curvearrowleft}{B}) = 0 = \mu_0 \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{X} \\ \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad &\Longrightarrow \quad -\mu_0 \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{X} = 0 \end{aligned}$$

$\vec{X}$ must be a function of $\vec{E}$ and $\overset{\curvearrowleft}{B}$. According to equation (3.1):

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} \quad \Longrightarrow \quad -\mu_0 \epsilon_0 \frac{\partial}{\partial t}, (\vec{\nabla} \cdot \vec{E}) + \vec{\nabla} \cdot \vec{X} = 0$$

The operators ∂_t and $\vec{\nabla}$ act on independent variables, they commute, so:

$$\vec{\nabla} \cdot \left(\vec{X} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) = 0$$

The general solution of this equation is:

$$\vec{X} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{Y}$$

This vector $\vec{Y}$ is a function of the fields. It must be linear in $\vec{E}$ and $\breve{\vec{B}}$(assumed sufficient to describe the physical situation). The vector $\vec{X}$ can only contain first-order derivatives, so $\vec{Y}$ must not contain derivatives of $\vec{E}$ and $\breve{\vec{B}}$:

$$\begin{aligned} & \vec{Y} = \alpha\vec{E} + \beta\breve{\vec{B}} \\ \Longrightarrow \quad & \vec{X} = \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \alpha\vec{\nabla}\cdot\vec{E} + \beta\vec{\nabla}\cdot\breve{\vec{B}} \end{aligned}$$

$\vec{X}$ is a true vector and $\vec{\nabla}\times\vec{E}$ a pseudovector. So $\alpha = 0$. This preserves invariance by reflection in a mirror: if one does an experiment on a laboratory table, and if one looks at it in a mirror, the image of the experiment must also be physically feasible. All other equations preserving this invariance, there is no reason to abandon this property[2]. We therefore have left:

$$\begin{aligned} & \vec{X} = \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \beta\vec{\nabla}\times\breve{\vec{B}} \\ \Longrightarrow \quad & (1-\beta)\vec{\nabla}\cdot\breve{\vec{B}} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \\ \Longleftrightarrow \quad & \vec{\nabla}\cdot\breve{\vec{B}} - \frac{\epsilon_0\mu_0}{1-\beta}\frac{\partial\vec{E}}{\partial t} = \frac{\mu_0}{1-\beta}\vec{J} \end{aligned}$$

Experimentally, one only has access to the combination $\frac{\epsilon_0\mu_0}{1-\beta}$ without being able to know what μ_0 and $1-\beta$ are worth separately. One therefore limits oneself to redefining μ_0 (renormalization procedure) and the solution:

$$\vec{X} = \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

is just as general. This is the term that Maxwell called "displacement current". For him, it was a true displacement of charges in the ether. For us, there is no ether and $\partial_t\vec{E}$ has nothing to do with a current. It is a field term that differs in no way from $\partial_t\breve{\vec{B}}$.

The correct writing of the four equations separates the field terms from the source terms:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \tag{3.3}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\breve{\vec{B}}}{\partial t} = 0 \tag{3.4}$$

$$\vec{\nabla}\cdot\breve{\vec{B}} = 0 \tag{3.5}$$

$$\vec{\nabla}\times\breve{\vec{B}} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \tag{3.6}$$

In particular when $\vec{J} = \vec{0}$, there exists a non-trivial solution to this system where the fields $\vec{E}$ and $\breve{\vec{B}}$ propagate. Equations (3.4) and (3.6) are symmetric. (3.4) translates the induction of an electric field by a variable magnetic field, and (3.6) the induction of a magnetic field by a variable electric field.

To return to the problem of the radioactive sphere, one can evaluate $\vec{J}$ on a sphere of radius r(fig. 3.3).

[2]Note that this invariance is *not* verified for weak interactions (beta radioactivity). But it is valid for the electromagnetic interactions that interest us here.

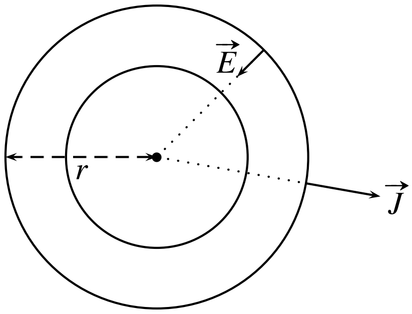


Fig. 3.3: Sphere of radius r

$$4\pi r^2 J = -\frac{\mathrm{d}Q}{\mathrm{d}t} \qquad \Longrightarrow \qquad J = -\frac{1}{4\pi r^2}, \frac{\mathrm{d}Q}{\mathrm{d}t}$$

The field created by a spherical charge Q at point M is:

$$E = \frac{1}{4\pi\varepsilon_0}, \frac{Q}{r^2} \qquad \Longrightarrow \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{4\pi\varepsilon_0 r^2}, \frac{\mathrm{d}Q}{\mathrm{d}t}$$

We therefore have:

$$-\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}$$

so $\vec{\nabla} \times \vec{B} = \vec{0}$ and, since $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{B} = \vec{0}$.
Maxwell's triumph was to show that in the absence of sources the system admitted non-trivial solutions in which E and B propagated with a speed

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

For strictly dimensional reasons, one can see that this term is indeed a speed:

$$(3.6) \qquad \Longrightarrow \qquad [\varepsilon_0 \mu_0] = \frac{[B]}{[E]} \frac{1}{L}$$

$$(3.4) \quad \Longrightarrow \quad \frac{[B]}{[E]} = \frac{T}{L} \quad \Longrightarrow \quad [\varepsilon_0 \mu_0] = \frac{T^2}{L^2} \quad \Longrightarrow \quad \left[\frac{1}{\varepsilon_0 \mu_0}\right] = \frac{L}{T}$$

So the propagation speed of electromagnetic waves must be $\frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ up to a coefficient, which can be anything. But according to the zero principle of physics, all coefficients are close to 1, here equal to 1.
The first to see that $\varepsilon_0 \mu_0$ and c were related is Riemann, in 1840. For him, ε_0 and μ_0 were constants of magnetostatics and electrostatics. He therefore posed the problem of a relation between electricity, magnetism and light. But his communication remained ignored.

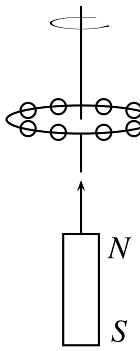
3.2 Hypothesis of magnetic monopoles

In equation (3.5)$\vec{\nabla} \cdot \vec{B} = 0$, the zero means that there is no magnetic charge. In 1930, Dirac asks the question of the possible existence of magnetic charges. To preserve the symmetry of the system of equations, it is necessary to modify both equations (3.5) and (3.4):

$$\vec{\nabla} \cdot \overset{\curvearrowleft}{B} = \sigma \quad \text{magnetic charge density}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \overset{\curvearrowleft}{B}}{\partial t} = \vec{g} \quad \text{magnetic charge current}$$

The magnetic charge is a pseudoscalar, that is, the image in a mirror is a magnetic charge of opposite sign. This theory has the interest of justifying the quantization of electric charge. Nothing else allows it. To understand the relation, consider the following paradox, due to Feynman (cf. Feynman 17-8):



We perform a thought experiment using a disk mounted on a vertical axis. On the periphery of the disk, we place metal balls that we electrically charge. We approach a magnet along the axis. There therefore exists at the level of the charges a variable magnetic field, therefore an electric field. Consequently, the disk starts to rotate. How is this compatible with the conservation of angular momentum?

One can think that this momentum comes from the magnet. To verify this, one can replace it with a current loop and note that no inverse phenomenon acts on the loop sufficiently to modify the angular momentum of the electrons.

The angular momentum actually comes from the electromagnetic field which carries energy, a density of momentum and *a density of angular momentum*. On the other hand, there is no angular momentum or momentum in a pure electric or magnetic field.

Let us return to the hypothesis of the existence of magnetic monopoles. Suppose that a magnetic monopole of intensity M and an electric charge Q are in the same place. This system creates an electric field and a magnetic field, and this *electromagnetic* field has angular momentum as well as energy[3] and momentum. However, angular momentum is quantized in units of $\hbar$, σ is proportional to MQ and we arrive, all calculations done, at:

$$MQ = n\hbar \quad \text{(in a suitable unit system.)}$$

Consequently, if there exists even one magnetic monopole in the entire universe, this imposes that all electric charges be quantized. Magnetic charges must also be.

Unfortunately, no magnetic charge has ever been found. They may be located in very particular sites; for example, if these charges are present in cosmic rays, they are guided by the Earth's magnetic field and must gather at the poles. Drillings have been done in Antarctica which have yielded no results.

[3] Cf "classical radius" of the electron.

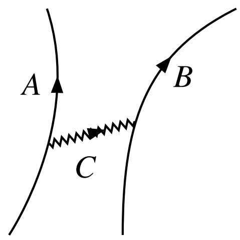
We have an upper limit on the density of magnetic charges in the universe, as well as a lower limit on their mass. It may be that they are very heavy and move very little, which would make them difficult to find[4].

Attempts have been made to imagine a relationship between quarks and monopoles, but without success.

3.3 Relations between electromagnetism and quantum mechanics

The photon represents the quantization of the electromagnetic field. It is a "particle" in the new sense of the term, that is, something other than an old particle or a wave.

One can simply show the relation between the particle associated with a force field, and the range of the force field in question. This goes far beyond the framework of electromagnetism. The first study was done by Yukawa around 1935 about nuclear forces. Let A and B be two objects that move by acting on each other through a force field that propagates. One can represent this field by a particle C and symbolically draw its displacement.



In fact C being a quantum particle, has no trajectory[5].
The quantum particle A therefore emits a particle C which is reabsorbed by B according to the process:

$$\begin{cases} A \to A + C \\ B + C \to B \end{cases}$$

This particle C has transported energy and momentum. So there is for A and B a $\frac{\mathrm{d}p}{\mathrm{d}t}$ which is indeed what we call a force.

However, such processes are strictly incompatible with the laws of conservation of energy and momentum. One cannot conserve both E and $\vec{p}$ if a particle disintegrates into itself and another. If momentum is conserved, energy cannot be conserved.
In fact there is indeed a variation ΔE in energy, because the value of the energy of A is not determined. There exists a dispersion in energy, a spectrum width ΔE. This ΔE is related to the characteristic evolution time of the system by

$$\Delta E \cdot \Delta t \approx \hbar$$

[4]Nothing proves, however, that they do not exist, and it would be very nice to discover them. It turns out that in recent years, physicists have invented quantities of particles, and unfortunately it is always the least interesting ones that seem to exist! On the other hand, the existence of magnetic charges would restore a superb symmetry in Maxwell's equations, which would be really elegant, assuming that the Good Lord has the same sense of elegance as Dirac. Instead of that, we only find quarks which are only a disheartening repetition of the decomposition into pure bodies, simple bodies, atoms, nuclei and electrons, protons and neutrons, etc.

[5]It is not that one cannot determine this trajectory, but simply that it does not exist

The system not being in a stationary state, it is not in an energy eigenstate, but occupies an energy band, of width ΔE, which allows the creation of particle C. For this, ΔE must be comparable to the mass energy of the intermediate particle (a few mc^2 at most to account for its kinetic energy):

$$\Delta E \sim mc^2 \implies \Delta t \sim \frac{\hbar}{mc^2}$$

Δt is the time necessary for the "exchange" of particle C. During this time, it can travel at most $c\Delta t$. The range of the force exerted by A on B is therefore:

$$a = c\Delta t = \frac{\hbar}{mc}$$

For a nuclear force, it had been observed that the range was on the order of 10^{-15} m. This corresponds for particle C to a mass on the order of 100 MeV. Yukawa therefore "invented" a particle of mass 100 MeV, baptized meson. Around 1945, they were believed to be found in cosmic rays; these particles (μ) had a correct mass but not the right properties. They should have interacted very strongly with nucleons, but the effective section of nucleon-meson μ interaction is very small. A consequence of this weakness is precisely the great abundance of cosmic muons at ground level. In fact, muons are only decay products of true mesons (π).

If we want to transpose this theory to the case of electromagnetism, we note that the smaller the mass of the exchange particle, the greater the range. At the limit, if m is zero, the range is infinite, which is the case for electromagnetic forces. The Yukawa potential of an interaction associated with a particle of mass m is, in first approximation:

$$U(r) = K, \frac{e^{-r/a}}{r} \quad \text{where} \quad a = \frac{\hbar}{mc}$$

The factor $\frac{1}{r}$ is due to the fact that space has 3 dimensions, but the factor $e^{-r/a}$ decreases much faster. The force becomes completely negligible beyond 2 or 3 a(2 or 3 fermi for nuclear forces).

On the contrary, a Coulombian potential of the form $U = \frac{K}{r}$ decreases very slowly, and one can consider its range as infinite. This observation allows testing the nullity of the photon's mass and the constancy of the speed of light. If the speed of light was not c(it is now a universal constant, constant of space-time), the mass of the photon would not be zero. The range of the electromagnetic force would therefore be finite, namely:

$$L = \frac{\hbar}{m_\gamma c}$$

To give an upper limit to m_γ, we must see on what distance scale the law $U = \frac{K}{r}$ is true.

$$U = \frac{K}{r} \quad \text{over a distance } L \implies a > L \implies m_\gamma < \frac{\hbar}{Lc}$$

A simple verification on $L \sim 1$ m gives:

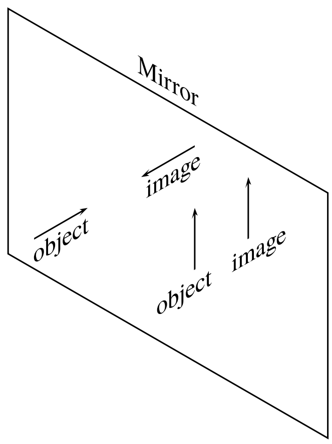
$$m\gamma < \frac{6{,}62 \times 10^{-34}}{2\pi \times 3 \times 10^8} \sim 3{,}5 \times 10^{-43} \text{ kg} = 3{,}5 \times 10^{-40} \text{ g}$$

The best limits on the photon's mass currently come from space studies; we measure, thanks to a satellite, the field of the Earth's magnetic dipole (which has the advantage of being much more important than the electric fields that can be accumulated in the laboratory). These measurements are made far enough from the Earth for the field to be almost purely dipolar (at 2 or 3 R_T[6]).

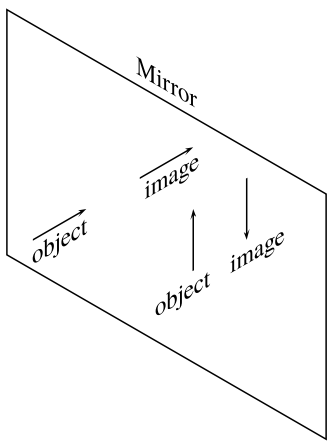
The distance L over which $U = \frac{K}{r}$ is verified is then on the order of R_T. The same measurements, repeated on Jupiter, give an even lower limit on m_γ. Any verification of Coulomb's law is therefore a verification of the nullity of the photon's mass, and consequently of the constancy of the speed of light.

3.4 Appendix

3.4.1 Reflection of a true vector



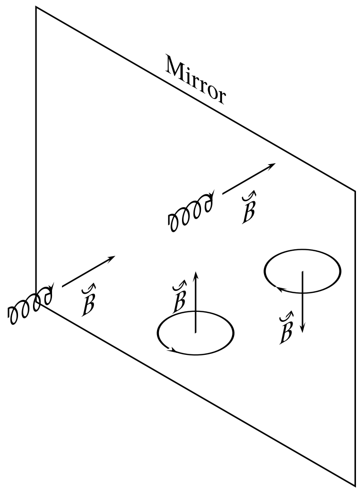
3.4.2 Reflection of a pseudovector



[6] R_T: Earth radius

Example: Field $\overset{\curvearrowleft}{B}$ created by

1. a solenoid
2. a loop



Chapter 4

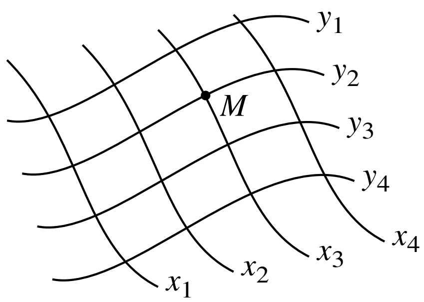
TWO DIFFERENT POINTS OF VIEW ON GENERAL RELATIVITY

(written by Anne Jaoul)

One must not confuse the historical order and the logical structure of things: Historically, Einstein (1910) tried to formulate the laws of physics in such a way that they are valid in any coordinate system. In the theory of special relativity, the laws were only valid in a particular class of reference frames: inertial reference frames. By wanting to geometrize the theory of relativity, so as to express these laws in a form independent of the coordinate system, Einstein was led to make a theory of gravitation, which is general relativity.

4.1 Geometric theory of general relativity

The law $\vec{\nabla} \cdot \overset{\leftrightarrow}{B} = 0$ is valid in any Euclidean reference frame. It would be interesting to formulate this law in a form such that it is valid in any coordinate system, even non-Cartesian, even if it has curved axes for example. For example, for a two-dimensional continuum, the coordinate system could consist of two families of absolutely arbitrary curves: an infinite set of curves x_i, infinitely close and not intersecting and another set of curves y_i, satisfying the same conditions. Each point in space is then located by a curve x_i and a curve y_i these two numbers being the coordinates of the point.



To express the laws of physics in any reference frame, it is necessary to use the tools of tensor algebra, and introduce an arbitrary metric which will no longer be:

$$\mathrm{d}\vec{\ell}^{\,2} = \mathrm{d}x^2 + \mathrm{d}y^2$$

(special case where the curves x_i and y_i are perpendicular in a two-dimensional Euclidean space and where (x, y) are the Cartesian coordinates of the point) We will then have a general form of the type

$$d\vec{\ell}^2 = \sum g_{ij}, \mathrm{d}x_i \mathrm{d}x_j$$

that is a bilinear form with respect to infinitesimal elements, g_{ij} depending on the point where one is. (For example when using polar coordinates in a plane we have a form of the type

$$d\vec{\ell}^2 = \mathrm{d}\vec{r}^2 + r^2 \mathrm{d}\theta^2$$

so $g_{11} = 1$, $g_{12} = 0$ and $g_{22} = r^2$)

To express the laws of physics in any reference system, it suffices to involve in these laws, the expression of the fundamental metric tensor (g_{ij}). One could do the same for Newtonian physics, it is a mathematical problem. But why does the desire to have a universal geometric expression of any law lead to a theory of gravitation?

One of the fundamental properties of gravitational forces is the equivalence of inertial mass and gravitational mass. Inertial mass (m_i) is a coefficient that characterizes the resistance of a body to a change in motion, hence a coefficient characterizing the dynamic properties of a body. Gravitational mass (m_g) is the source quantity of the gravitational field, just as electric charge is the source of the electromagnetic field. However, it turns out that there is universal proportionality between m_i and m_g. By choosing units appropriately, one can confuse them, but *a priori*, one has no right to say that it is the same thing. Newton's law is written:

$$F = K \frac{m_g m'_g}{z^2}$$

Experimentally $m_g = \gamma m_i$ where γ is a universal constant independent of the body in question, hence

$$F = \gamma^2 K \frac{m_i m'_i}{z^2}$$

This equality of the two masses m_i and m_g is completely accidental in classical mechanics and plays no role in its structure. In the theory of general relativity, this equality is fundamental because of one of its physical consequences: the trajectory of an object in the gravitational field does not depend on its mass. Indeed, according to the fundamental relation of dynamics

$$m_i \vec{\gamma} = m_g \vec{g} \quad \text{hence} \quad \vec{\gamma} = \vec{g} \quad \text{if} \quad m_i = m_g$$

With identical initial data, the trajectories will be identical: when launching a satellite, its trajectory depends on the launch point and initial velocity but not on its mass.

The fundamental consequence of the equality of inertial and gravitational masses is that in a gravitational field, the trajectories not depending on the mass of the body considered, present themselves as having purely geometric properties. It is because of this specific property of the field that it is interesting to give a completely geometrized description of it. One could do the same for electric forces but the relation $m\vec{\gamma} = q\vec{E}$ shows that the trajectories depend on the physical coefficient $\frac{q}{m}$ and will not be identical, under the same initial conditions, for different bodies. The geometrized writing of the equations is possible but uninteresting since it depends on a physical factor, whereas in the case of gravitational forces, the trajectories are purely geometric properties of the field. One can then forget the physical properties to treat only the mathematical problem. General relativity then takes the form of a purely geometric theory.

4.2 Theory of the gravitational field: physical point of view

However, one can approach the problem from another, more physical point of view: indeed, this geometrized conception of gravitation has disadvantages. First, it separates gravitational interactions from other fundamental interactions (strong, weak, electromagnetic). It would be more satisfactory to try to treat them on the same plane. Moreover, once general relativity is written in a completely geometrized form, we obtain a rigid theory that can no longer be modified. If, for example, the equivalence principle, inertial mass=gravitational mass is only approximately true, and if a correction must be made to the theory, one cannot do it in the geometric framework, too rigid. It would be necessary to develop a slightly different point of view, so as not to start by imposing the equality $m_i = m_g$, but to make a theory of the gravitational field capable of evolving with knowledge.

For this, it suffices to follow the same approach as for the electromagnetic field. What is gravitation? It is a point interaction transmitted by a field. A mass creates around it a gravitational field that propagates and acts on another mass.

But this field must be described in accordance with special relativity. We know that the mathematical being that will represent it must have particular geometric properties in space-time and must be a quadrivector or a tensor of a certain kind. We therefore seek a form of scalar or tensor potential, as simple as possible, and by building axiomatically a relativistic Einsteinian theory, we try to obtain a theory of gravitation that conforms to the three standard tests of general relativity:

- explain the redshift of the frequencies of radiations emitted by stars
- explain the deflection of light rays in a gravitational field
- account for the displacement of Mercury's perihelion (orbits are not rigorously ellipses but non-closed curves)

One could have a scalar potential (or order 0 tensor), this is the example of the Newtonian field in which the field derives from a scalar potential. Such a theory is possible, but cannot account for the displacement of Mercury's perihelion.

The electromagnetic field derives from a more complicated potential: it is a quadrivector (or order 1 tensor)$(V, \vec{A})$. But the properties of the electromagnetic field are different from those of the gravitational field.

It is therefore necessary to try an order 2 tensor $(G_{\mu\nu})$ and one can build a field theory, axiomatically as Landau and Lifchitz did for the electromagnetic field (through a Lagrangian). But to fix the theory, it is necessary to make assumptions about the nature of the sources and research how the field is coupled to its sources. This is the problem of terms in ρ, $\vec{j}$ in electromagnetic theory, where the sources of the field are only electric charges.

In the gravitational field, we make the assumption that the source of the field is mass. But in Einstein's theory of relativity, there is equivalence between the mass and the energy of a system: the mass of a system is equal to the energy in the inertial reference frame (where momentum is zero). It is therefore necessary to generalize and take as source of the gravitational field all the energy of a system. Hence a specific character of the gravitational field: this one transports energy which is itself a source of field. In other words, we have a self-coupling of the gravitational field: its energy contributes to creating the field itself. We would have the same situation in the electromagnetic field if the photon was a charged particle: not only would it be a vector of the field but moreover it would generate the field itself.

The consequence of all this is that the equations of the gravitational field will not be linear:

- in the electromagnetic field, the field $\vec{D}$ and the source are related by a relation $\vec{\nabla} \cdot \vec{D} = \rho$
- in the gravitational field, we have a relation of the type

$$aG = T + \text{term in } G \quad \text{(where } a \text{ is a differential operator)}$$

T being the density of matter at the point considered and the term in G introducing a non-linear quantity since the energy density is a quadratic quantity relative to the field.

The fact that the field itself is a source of field through the energy it transports, therefore leads to very complicated theories because non-linear and this for physical and not formal reasons. Consequently:

- the differential equations not being linear, there is no general mathematical theory to solve them. Approximations must be made to linearize them
- the superposition principle does not apply
- the notion of gravitational waves is complicated because the equations not being linear, it must be ensured that they effectively represent propagation phenomena. They will not at all have the same characters as electromagnetic waves

4.3 Comparison of the two points of view

What relations are there between this relativistic theory of gravitation and the theory of general relativity as a geometrized theory?

The relativistic theory of gravitation does not have a fundamentally different physical interpretation from electromagnetic theory: in the flat Minkowski space of special relativity, we have a field $\vec{G}(\vec{r}, t)$. Suppose we want to verify that this space is indeed flat; we would have to do geometry and use measuring instruments. But all objects are coupled to the gravitational field (this was not the case in the electromagnetic field).

Take for example the standard meter in a flat space, where the gravitational field $\vec{G}(\vec{r}, t)$ reigns. It will undergo the action of $\vec{G}$ in a universal way, that is, independent of its mass. Under the effect of $\vec{G}$, the measurement result, which in a flat space would be 1 m, will be larger because of the shortening of the standard meter. But we will have the same effect on all measuring instruments and consequently, we will not be able to highlight it. We will only study the *apparent* geometric properties of space, which will depend on the value at this point of the gravitational field; one can always assume that the underlying space is a flat space; the spatial properties being universally affected by the gravitational field, everything happens as if we were in a twisted space, curved, which instead of being constant pseudo-Euclidean, has a variable metric.

In fact, this tensor field $G_{\mu\nu}(\vec{r}, t)$ will play the role of the apparent metric. In other words, the coefficient g_{ij} which from the standard point of view of general relativity appears only as a geometric property, from the point of view of the relativistic theory of gravitation will appear as being the gravitational field, symmetric order 2 tensor with ten components.

We therefore have two theories: one physical which is a field theory and the other a purely geometric theory.

On the formal plane, they are absolutely equivalent; the advantage of the physical point of view being to be able to compare the gravitational field to the electromagnetic field. The development of the theory of the two fields follows the same approach, but they differ by this universal coupling which makes that effectively everything happens as if the theory of the gravitational field was a completely geometrized theory.

Another advantage of the physical point of view is that, if the theory is false (as is probably the case) it will be easier to modify it, than from the geometric point of view, which is that of traditional general relativity: the coupling is perhaps not universal, there may be unknown forms of matter that would not be coupled to the gravitational field, etc. In fact, there are already theories of gravitation more general than the standard theory.

The geometric point of view is advantageous for those who do a lot of differential geometry because mathematically it is much simpler.

Let us conclude to finish that general relativity is less general than special relativity. Indeed, it is special relativity that is the general theory of space-time structure and that imposes its constraints on all phenomena that take place in this space-time.

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