TANGENT AND OSCULATING SPACES

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Abstract. Two spaces whose metrics have the same coefficients at a point are said to be *tangent* at that point. Two spaces whose metrics have the same coefficients at a point and the same derivatives of these coefficients are said to be *osculating* at that point. This notion does not apply to Euclidean spaces (of which the line and the plane are examples). They can be tangent but cannot be osculating.

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1 In two dimensions

1.1 Tangent line to a parabola

Example 1.1. *Let (o, x, y) be a Cartesian coordinate system in the two-dimensional Euclidean space (the Euclidean plane). The square of the length element is written $ds^2 = dx^2 + dy^2$. Consider the parabola with equation $y = x^2$. By differentiating:*

$$dy = 2xdx$$

If the length element belongs to the parabola:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= dx^2 + 4x^2dx^2 \\ &= \left(4x^2 + 1\right) dx^2 \end{aligned}$$

The metric tensor of the parabola has only one element:

$$g_{xx} = 4x^2 + 1$$

Note that this is also the metric tensor of the parabola with equation $y = -x^2$. Consider the line with equation $y = ax + b$ (one-dimensional Euclidean space). By differentiating:

$$dy = adx$$

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If the length element belongs to the line:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= dx^2 + a^2 dx^2 \\ &= \left(a^2 + 1\right) dx^2 \end{aligned}$$

The metric tensor of the line has only one element:

$$g_{xx} = a^2 + 1$$

If at a point $M(x_0, y_0)$ belonging to two curves the metrics are equal, then the curves are tangent at that point. The lines tangent at $M(x_0, y_0)$ to the parabola are such that the metrics are equal at that point:

$$\begin{aligned} a^2 + 1 &= 4x_0^2 + 1 \\ a &= \pm 2x_0 \end{aligned}$$

We keep the positive sign because the negative sign corresponds to the lines tangent to the parabola with equation $y = -x^2$:

$$a = 2x_0$$

The point $M(x_0, y_0)$ belongs to the parabola, $y_0 = x_0^2$, and to the line $y_0 = ax_0 + b$:

$$\begin{aligned} b &= y_0 - ax_0 \\ &= x_0^2 - 2x_0^2 \\ &= -x_0^2 \end{aligned}$$

The equation of the tangent at $M(x_0, y_0)$ to the parabola is written:

$$y = 2x_0 x - x_0^2$$

For example at the point $(1, 1)$ of the parabola:

$$y = 2x - 1$$

If one tries to find the line osculating the parabola, one obtains,

$$\begin{aligned} \frac{\partial}{\partial x}\left(a^2 + 1\right)\Big|_{x_0} &= \frac{\partial}{\partial x}\left(4x^2 + 1\right)\Big|_{x_0} \\ 0 &= 8x_0 \\ x_0 &= 0 \end{aligned}$$

which does not make sense.

1.2 Circles tangent to a parabola

Example 1.2. *Consider the circle with center (x_c, y_c) and radius r, with equation:*

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

By differentiating:

$$\begin{aligned} 2(x - x_c)\,dx + 2(y - y_c)\,dy &= 0 \\ dy &= \frac{-(x - x_c)}{y - y_c}\,dx \\ dy^2 &= \frac{(x - x_c)^2}{(y - y_c)^2}\,dx^2 \end{aligned}$$

If the length element belongs to the circle:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= dx^2 + \frac{(x-x_c)^2}{(y-y_c)^2}\, dx^2 \\ &= \left[\frac{(x-x_c)^2}{(y-y_c)^2} + 1\right] dx^2 \end{aligned}$$

The metric tensor of the circle has only one element:

$$\begin{aligned} g_{xx} &= \frac{(x-x_c)^2}{(y-y_c)^2} + 1 \\ &= \frac{(x-x_c)^2 + (y-y_c)^2}{(y-y_c)^2} \\ &= \frac{r^2}{r^2 - (x-x_c)^2} \end{aligned}$$

At the point of tangency $M(x_0, y_0)$ the metrics are equal:

$$\begin{aligned} 4x_0^2 + 1 &= \frac{(x_0-x_c)^2}{(y_0-y_c)^2} + 1 \\ 4x_0^2\,(y_0-y_c)^2 &= (x_0-x_c)^2 \\ (y_0-y_c)^2 &= \left(\frac{x_0-x_c}{2x_0}\right)^2 \\ y_0 - y_c &= \pm\frac{x_0-x_c}{2x_0} \\ y_c &= y_0 \mp \frac{x_0-x_c}{2x_0} \end{aligned}$$

We keep the positive sign because the negative sign corresponds to the circles tangent to the parabola with equation $y = -x^2$:

$$y_c = y_0 + \frac{x_0-x_c}{2x_0} \tag{1}$$

Let us check if the fact that the point M belongs to both curves provides a new equation. Let us redo the calculation using the radius:

$$\begin{aligned} 4x_0^2 + 1 &= \frac{r^2}{r^2 - (x_0-x_c)^2} \\ 4x_0^2 r^2 - \left(4x_0^2+1\right)(x_0-x_c)^2 &= 0 \\ r^2 &= \frac{4x_0^2+1}{4x_0^2}\,(x_0-x_c)^2 \\ r &= \pm\frac{\sqrt{4x_0^2+1}}{2x_0}\,(x_0-x_c) \end{aligned}$$

The point $M(x_0, y_0)$ belongs to the parabola and to the circle:

$$(x_0 - x_c)^2 + (y_0 - y_c)^2 = r^2$$

$$\begin{aligned} x_0^2 - y_c &= \pm\sqrt{r^2 - (x_0 - x_c)^2} \\ y_c &= x_0^2 \mp \sqrt{r^2 - (x_0 - x_c)^2} \\ &= x_0^2 \mp \sqrt{\frac{4x_0^2+1}{4x_0^2}(x_0 - x_c)^2 - (x_0 - x_c)^2} \\ &= x_0^2 \mp \sqrt{\frac{1}{4x_0^2}(x_0 - x_c)^2} \\ &= x_0^2 \mp \frac{x_0 - x_c}{2x_0} \end{aligned}$$

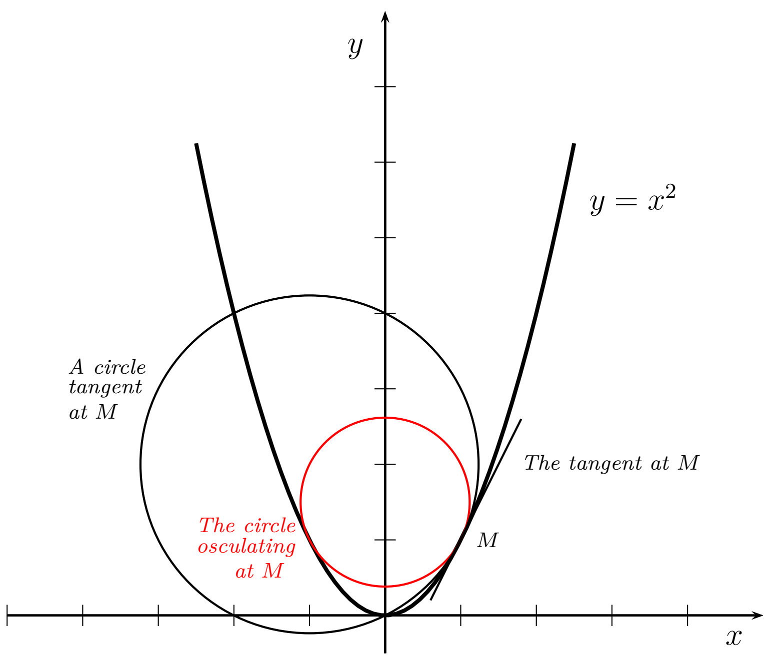
We recover equation (1). We therefore have only one equation for the two unknowns x_c, y_c. While there is only one tangent line at any point on the parabola, there is an infinity of circles tangent at each point $M(x_0, y_0)$ of the parabola. The centers of these circles constitute the line perpendicular to the parabola at the considered point. For example at the point $(1, 1)$ of the parabola,

$$r = \pm\frac{\sqrt{5}}{2}(1 - x_c) \qquad \text{and} \qquad y_c = 1 + \frac{1 - x_c}{2}$$

If we take $x_c = -1$,

$$r = \sqrt{5} \qquad \text{and} \qquad y_c = 2$$

1.3 Osculating circle to a parabola



Example 1.3. *Consider the circle with center* (x_c, y_c) *and radius* r*, with equation:*

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

osculating the parabola with equation $y = x^2$*. We then always have the equality of the metrics at the point* $M(x_0, y_0)$ *which gives,*

$$(y_0 - y_c)^2 = \left(\frac{x_0 - x_c}{2x_0}\right)^2$$

The circle being osculating the parabola, we also have the equality of the derivatives of the metrics at the point $M(x_0, y_0)$*:*

$$\begin{aligned}
\frac{\partial}{\partial x}\left(4x^2 + 1\right)\Big|_{x_0} &= \frac{\partial}{\partial x}\left[\frac{(x - x_c)^2}{(y - y_c)^2} + 1\right]_{x_0} \\
8x_0 &= \frac{2\,(x_0 - x_c)}{(y_0 - y_c)^2} \\
(y_0 - y_c)^2 &= \frac{x_0 - x_c}{4x_0}
\end{aligned}$$

So that,

$$\begin{aligned}
\left(\frac{x_0 - x_c}{2x_0}\right)^2 &= \frac{x_0 - x_c}{4x_0} \\
x_0 - x_c &= x_0 \\
x_c &= 0
\end{aligned}$$

The center of the osculating circle to the parabola is on the y-axis, and its y-coordinate is:

$$\begin{aligned}
y_c &= y_0 + \frac{x_0 - x_c}{2x_0} \\
&= y_0 + \tfrac{1}{2}
\end{aligned}$$

For example at the point $(1, 1)$ *the center of the osculating circle has coordinates* $(0; 1.5)$ *and radius:*

$$\begin{aligned}
r^2 &= (x_0 - x_c)^2 + (y_0 - y_c)^2 \\
&= 1^2 + (1 - 1.5)^2 = \frac{5}{4} \\
r &= \frac{\sqrt{5}}{2} \approx 1.118
\end{aligned}$$

2 In three dimensions

2.1 Tangent plane to a paraboloid of revolution

Example 2.1. *Let* (o, x, y, z) *be a Cartesian coordinate system in the three-dimensional Euclidean space. The square of the length element is written* $ds^2 = dx^2 + dy^2 + dz^2$*. Consider the paraboloid of revolution with equation* $z = x^2 + y^2$*. By differentiating:*

$$dz = 2xdx + 2ydy$$

If the length element belongs to the paraboloid of revolution:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dx^2 + dy^2 + 4x^2dx^2 + 4y^2dy^2 + 8xydxdy \\ &= \left(4x^2+1\right)dx^2 + \left(4y^2+1\right)dy^2 + 8xydxdy \end{aligned}$$

The metric tensor of the paraboloid of revolution has elements:

$$\begin{cases} g_{xx} = 4x^2+1 \\ g_{yy} = 4y^2+1 \\ g_{xy} = g_{yx} = 4xy \end{cases}$$

Note that this is also the metric tensor of the paraboloid of revolution with equation $z=-x^2-y^2$. Consider the Euclidean plane with equation $ax+by+cz+d=0$ (two-dimensional Euclidean space). By differentiating:

$$adx + bdy + cdz = 0$$

If the length element belongs to the plane:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dx^2 + dy^2 + \frac{1}{c^2}\left(-adx - bdy\right)^2 \\ &= \left(\frac{a^2}{c^2}+1\right)dx^2 + \left(\frac{b^2}{c^2}+1\right)dy^2 + \frac{2ab}{c^2}dxdy \end{aligned}$$

The metric tensor of the plane has elements:

$$\begin{cases} g_{xx} = \dfrac{a^2}{c^2}+1 \\ g_{yy} = \dfrac{b^2}{c^2}+1 \\ g_{xy} = g_{yx} = \dfrac{ab}{c^2} \end{cases}$$

The planes tangent at $M(x_0,y_0,z_0)$ to the paraboloid of revolution are such that the metrics are equal at that point:

$$\begin{cases} \dfrac{a^2}{c^2}+1 = 4x_0^2+1 \\ \dfrac{b^2}{c^2}+1 = 4y_0^2+1 \\ \dfrac{ab}{c^2} = 4x_0y_0 \end{cases} \Rightarrow \begin{cases} a = \pm 2cx_0 \\ b = \pm 2cy_0 \\ ab = 4c^2x_0y_0 \end{cases}$$

The first two equalities give $ab = \pm 4c^2x_0y_0$. To conform to the third equality, we must have the following condition: If $a = 2cx_0$ then $b = 2cy_0$, and if $a = -2cx_0$ then $b = -2cy_0$. We keep the negative sign because the positive sign corresponds to the planes tangent to the paraboloid of revolution with equation $z = -x^2 - y^2$:

$$\begin{cases} a = -2cx_0 \\ b = -2cy_0 \end{cases}$$

The point $M(x_0,y_0,z_0)$ belongs to the paraboloid of revolution,

$$z_0 = x_0^2 + y_0^2$$

and to the plane:

$$\begin{aligned} ax_0 + by_0 + cz_0 + d &= 0 \\ -2cx_0^2 - 2cy_0^2 + cz_0 + d &= 0 \\ -2cz_0 + cz_0 + d &= 0 \\ d &= cz_0 \end{aligned}$$

We then have:

$$\begin{cases} a = -2cx_0 \\ b = -2cy_0 \\ d = cz_0 \end{cases}$$

The equation of the tangent plane at $M(x_0, y_0, z_0)$ has equation:

$$\begin{aligned} -2cx_0x - 2cy_0y + cz + cz_0 &= 0 \\ -2x_0x - 2y_0y + z + z_0 &= 0 \end{aligned}$$

For example, the tangent plane to the paraboloid of revolution at the point $(1, 1, 2)$ has equation: $-2x - 2y + z + 2 = 0$. At each point of the paraboloid of revolution there is only one tangent plane.

2.2 Spheres tangent to a paraboloid of revolution

Example 2.2. *Consider the sphere with center (x_c, y_c, z_c) and radius r, with equation:*

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

By differentiating:

$$\begin{aligned} &2(x - x_c)\,dx + 2(y - y_c)\,dy + 2(z - z_c)\,dz = 0 \\ &dz = \frac{-(x - x_c)}{z - z_c}\,dx - \frac{(y - y_c)}{z - z_c}\,dy \\ &dz^2 = \frac{(x - x_c)^2}{(z - z_c)^2}\,dx^2 + \frac{(y - y_c)^2}{(z - z_c)^2}\,dy^2 + \frac{2(x - x_c)(y - y_c)}{(z - z_c)^2}\,dxdy \end{aligned}$$

If the length element belongs to the sphere:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dx^2 + dy^2 + \frac{(x - x_c)^2}{(z - z_c)^2}\,dx^2 + \frac{(y - y_c)^2}{(z - z_c)^2}\,dy^2 + \frac{2(x - x_c)(y - y_c)}{(z - z_c)^2}\,dxdy \\ &= \left[\frac{(x - x_c)^2}{(z - z_c)^2} + 1\right]dx^2 + \left[\frac{(y - y_c)^2}{(z - z_c)^2} + 1\right]dy^2 + \frac{2(x - x_c)(y - y_c)}{(z - z_c)^2}\,dxdy \end{aligned}$$

The metric tensor of the sphere has elements:

$$\begin{cases} g_{xx} = \dfrac{(x - x_c)^2}{(z - z_c)^2} + 1 \\ g_{yy} = \dfrac{(y - y_c)^2}{(z - z_c)^2} + 1 \\ g_{xy} = g_{yx} = \dfrac{(x - x_c)(y - y_c)}{(z - z_c)^2} \end{cases}$$

The spheres tangent at $M(x_0, y_0, z_0)$ to the paraboloid of revolution are such that the metrics are equal at that point:

$$\begin{cases} \dfrac{(x_0 - x_c)^2}{(z_0 - z_c)^2} + 1 = 4x_0^2 + 1 \\ \dfrac{(y_0 - y_c)^2}{(z_0 - z_c)^2} + 1 = 4y_0^2 + 1 \\ \dfrac{(x_0 - x_c)(y_0 - y_c)}{(z_0 - z_c)^2} = 4x_0 y_0 \end{cases} \Rightarrow \begin{cases} (x_0 - x_c)^2 = 4x_0^2 (z_0 - z_c)^2 \\ (y_0 - y_c)^2 = 4y_0^2 (z_0 - z_c)^2 \\ (x_0 - x_c)(y_0 - y_c) = 4x_0 y_0 (z_0 - z_c)^2 \end{cases}$$

$$\begin{cases} x_0 - x_c = \pm 2x_0 (z_0 - z_c) \\ y_0 - y_c = \pm 2y_0 (z_0 - z_c) \\ (x_0 - x_c)(y_0 - y_c) = 4x_0 y_0 (z_0 - z_c)^2 \end{cases}$$

Thus $x_0 - x_c$ and $y_0 - y_c$ have the same sign. We keep the negative sign:

$$\begin{cases} x_c = x_0 + 2x_0 (z_0 - z_c) \\ y_c = y_0 + 2y_0 (z_0 - z_c) \end{cases} \Rightarrow \begin{cases} x_c = x_0[1 + 2(z_0 - z_c)] \\ y_c = y_0[1 + 2(z_0 - z_c)] \end{cases}$$

We have only two equations for three unknowns x_c, y_c, z_c. There is therefore an infinity of spheres tangent at each point of the paraboloid of revolution. The centers of these spheres constitute the line perpendicular to the paraboloid at the concerned point. For example at the point $M(1, 0, 1)$ we have:

$$\begin{cases} x_c = 3 - 2z_c \\ y_c = 0 \end{cases}$$

If we take $x_c = -1$ the center of the tangent sphere has coordinates $(-1, 2, 0)$. We recover what we found in 2D with the parabola and the circle.

2.3 Osculating sphere to a paraboloid of revolution

Example 2.3. *Consider the sphere with center (x_c, y_c, z_c) and radius r, with equation:*

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2$$

osculating the paraboloid of revolution with equation $z = x^2 + y^2$. We always have the equality of the metrics at the point $M(x_0, y_0, z_0)$ which gives,

$$\begin{cases} x_c = x_0[1 + 2(z_0 - z_c)] \\ y_c = y_0[1 + 2(z_0 - z_c)] \end{cases}$$

and the equality of the derivatives of the metrics at the point $M(x_0, y_0)$:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(4x^2 + 1\right)_M = \dfrac{\partial}{\partial x}\left[\dfrac{(x - x_c)^2}{(z - z_c)^2} + 1\right]_M \\ \dfrac{\partial}{\partial y}\left(4y^2 + 1\right)_M = \dfrac{\partial}{\partial y}\left[\dfrac{(y - y_c)^2}{(z - z_c)^2} + 1\right]_M \\ \left.\dfrac{\partial 4xy}{\partial x}\right|_M = \left.\dfrac{\partial}{\partial x}\,\dfrac{(x - x_c)(y - y_c)}{(z - z_c)^2}\right|_M \\ \left.\dfrac{\partial 4xy}{\partial y}\right|_M = \left.\dfrac{\partial}{\partial y}\,\dfrac{(x - x_c)(y - y_c)}{(z - z_c)^2}\right|_M \end{cases} \Rightarrow \begin{cases} 4x_0 = \dfrac{x_0 - x_c}{(z_0 - z_c)^2} \\ 4y_0 = \dfrac{y_0 - y_c}{(z_0 - z_c)^2} \\ 4y_0 = \dfrac{y_0 - y_c}{(z_0 - z_c)^2} \\ 4x_0 = \dfrac{x_0 - x_c}{(z_0 - z_c)^2} \end{cases}$$

$$\begin{cases} 4x_0\left(z_0 - z_c\right)^2 = x_0 - x_c \\ 4y_0\left(z_0 - z_c\right)^2 = y_0 - y_c \end{cases} \Rightarrow \begin{cases} x_c = x_0\left[1 - 4\left(z_0 - z_c\right)^2\right] \\ y_c = y_0\left[1 - 4\left(z_0 - z_c\right)^2\right] \end{cases}$$

$$\begin{cases} x_0\left[1 - 4\left(z_0 - z_c\right)^2\right] = x_0[1 + 2(z_0 - z_c)] \\ y_0\left[1 - 4\left(z_0 - z_c\right)^2\right] = y_0[1 + 2(z_0 - z_c)] \end{cases}$$

These equalities being similar:

$$-4\left(z_0 - z_c\right)^2 = 2(z_0 - z_c) \quad \Rightarrow \quad z_0 - z_c = -\tfrac{1}{2} \quad \Rightarrow \quad z_c = z_0 + \tfrac{1}{2}$$

We then have:

$$\begin{cases} x_c = x_0\left[1 + 2\left(z_0 - z_0 - \tfrac{1}{2}\right)\right] \\ y_c = y_0\left[1 + 2\left(z_0 - z_0 - \tfrac{1}{2}\right)\right] \end{cases} \Rightarrow \begin{cases} x_c = 0 \\ y_c = 0 \end{cases}$$

The center of the osculating sphere is on the z-axis. We recover what we found in two dimensions with the parabola and the circle.

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