SCHRÖDINGER'S EQUATION

OLIVIER CASTÉRA

ABSTRACT. Schrödinger's equation is derived using the Planck-Einstein and de Broglie relations, and the conservation of mechanical energy. It therefore belongs to the domain of non-relativistic quantum mechanics.

CONTENTS

A Planck-Einstein and de Broglie Relations 1
B Dynamics of the Corpuscle 2
C Associated Wave Function 2
C.1 Sinusoidal Wave 2
C.2 Spatial Period 3
C.3 Temporal Period 4
C.4 Harmonic Oscillator 5
C.5 Wave Function 5
D Complex Notation 6
E The Harmonic Plane Wave 7
F The Wave Packet 8
G Group Velocity 9
H Dynamics of the Wave Packet 10
I Appendices 11
I.1 Propagation Equation 11
I.2 Complex Notation 12
I.3 Wave Packet 13
I.4 Phase Velocity 14

A Planck-Einstein and de Broglie Relations

The Planck-Einstein and de Broglie relations show the existence of a transition between corpuscular notions, energy E and momentum $\mathbf{p}$, and wave notions, pulsation ω and wave vector $\mathbf{k}$,

$$\begin{cases} E = \hbar\omega \quad \text{Planck-Einstein relation} & \text{(1a)} \\ \mathbf{p} = \hbar\mathbf{k} \quad \text{de Broglie relation} & \text{(1b)} \end{cases}$$

where $\hbar \overset{\text{def}}{=} h/(2\pi)$ is the quantum of action h called Planck's constant, divided by 2π, with:

$$h \overset{\text{def}}{=} 6{,}626\,070\,15 \times 10^{-34}\,\text{J/s}$$

Date: December 14, 2025.

It is the presence of the quantum of action that makes these relations belong to the domain of quantum mechanics. We will see that the energy E is both the energy of a wave and the mechanical energy of a corpuscle. These relations lead us to assume that what we will call a quanton can be modeled either as a corpuscle or as a wave. However, recent experiments on the Franson interferometer show that in fact these two models are not sufficient to model a quanton[1].

B Dynamics of the Corpuscle

In non-relativistic classical mechanics, the equation of the dynamics of the corpuscle associated with the quanton is given by the fundamental relation of dynamics (FRD),

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

in which each force $\mathbf{F}$ exerted on the corpuscle must be replaced by a force model representative of the situation. We can express the equation of the dynamics of the corpuscle in terms of the total mechanical energy E_m and the momentum $\mathbf{p}$ thanks to the conservation of mechanical energy, which is equivalent to the FRD when the force models all derive from a potential energy[2]. We then have

$$\boxed{E_m = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}, t)} \tag{2}$$

where $\mathbf{p}^2/(2m)$ is the kinetic energy, and $V(\mathbf{r}, t)$ is the potential energy from which the sum of forces derives. The goal is to find an equation of dynamics, called wave equation, for the wave associated with this same quanton.

C Associated Wave Function

C.1 Sinusoidal Wave

The circular functions sine and cosine are used to build a model of a sinusoidal wave. Let's take the cosine function :

$$f(x) = \cos x$$

According to appendix I.1 p. 11, the argument of any function must be of the form $(x - vt)$ for propagation along increasing x with speed v :

$$f(x, t) = f(x - vt)$$

The representation of the function

$$\psi(x, t) = \cos(x - v_\varphi t)$$

gives the illusion of wave propagation along the positive abscissa axis at the *phase velocity* v_φ of the wave. In fact, these are points oscillating vertically independently of each other, no

[1] Valerio Scarani *Initiation to Quantum Physics.* Vuibert, 2003.

[2] See Classical Mechanics.pdf

information is transmitted from one point to another. The amplitude of ψ (its value at a given instant) is between 1 and -1. To remove this limitation in our model of periodic wave, we introduce the *maximum amplitude* ψ_{max} of the motion :

$$\psi(x,t) = \psi_{max}\cos(x - v_\varphi t)$$

The argument of any function must be dimensionless otherwise the value of the function would depend on the chosen unit (for example meter or foot). Therefore x and t are here two dimensionless variables, and v_φ is a dimensionless parameter. To give them a physical dimension we introduce the *wave number* k homogeneous to the inverse of a length so that x can be a length. The equation modeling the periodic wave becomes :

$$\psi(x,t) = \psi_{max}\cos[k(x - v_\varphi t)]$$

By setting ω the *pulsation* or *angular frequency* of the wave such that,

$$\boxed{v_\varphi \stackrel{\text{def}}{=} \frac{\omega}{k}}$$

the equation is written :

$$\psi(x,t) = \psi_{max}\cos(kx - \omega t)$$

For ψ to remain a sinusoid, k and ω, and thus v_φ, must be constants in space and time. Since there is no reason for $\psi(0,0)$ to be equal to ψ_{max} at the origin of time and space, we introduce the constant φ_0 such that

$$\psi(x,t) = \psi_{max}\cos(kx - \omega t + \varphi_0) \tag{3}$$

where

$$\varphi = kx - \omega t + \varphi_0$$

is the *phase* of the sinusoidal motion, φ_0 is the phase at the origin of time and space.

C.2 Spatial Period

$\psi(x,t)$ being a cosine function, it is periodic in space,

$$\begin{aligned}\psi_{max}\cos(kx - \omega t) &= \psi_{max}\cos(kx - \omega t + 2\pi)\\ &= \psi_{max}\cos[k(x + 2\pi/k) - \omega t]\end{aligned}$$

so that

$$\begin{aligned}\psi(x,t) &= \psi\left(x + 2\pi/k, t\right)\\ &= \psi\left(x + \lambda, t\right)\end{aligned}$$

where the spatial period of the wave

$$\lambda \stackrel{\text{def}}{=} 2\pi/k$$

is called the *wavelength*.

C.3 Temporal Period

Similarly, $\psi(x,t)$ is periodic in time,

$$\begin{aligned}\psi_{max}\cos(kx-\omega t) &= \psi_{max}\cos(kx-\omega t+2\pi)\\ &= \psi_{max}\cos[kx-\omega(t+2\pi/\omega)]\end{aligned}$$

so that

$$\begin{aligned}\psi(x,t) &= \psi\left(x,t+2\pi/\omega\right)\\ &= \psi\left(x,t+T\right)\end{aligned}$$

where the temporal period of the wave is simply called the *period* of the wave :

$$T \stackrel{\text{def}}{=} 2\pi/\omega$$

We can write the wave equation (3) p. 3 in the form :

$$\begin{aligned}\psi(x,t) &= \psi_{max}\cos(kx-\omega t+\varphi_0)\\ &= \psi_{max}\cos\left(\frac{2\pi}{\lambda},x-\frac{2\pi}{T},t+\varphi_0\right)\\ &= \psi_{max}\cos\left[2\pi\left(\sigma x-\nu t+\varphi_0\right)\right]\end{aligned}$$

where

$$\sigma \stackrel{\text{def}}{=} 1/\lambda$$

is the *spatial frequency* of the wave, and where the temporal frequency of the wave is simply called the *frequency* of the wave :

$$\nu \stackrel{\text{def}}{=} 1/T$$

In summary, for the spatial part we have :

$$\begin{cases}k = 2\pi/\lambda = 2\pi\sigma\\ \lambda = 2\pi/k = 1/\sigma\\ \sigma = 1/\lambda = k/(2\pi)\end{cases}$$

We then measure the wavelength λ or the spatial frequency σ and calculate k. For the temporal part :

$$\begin{cases}\omega = 2\pi/T = 2\pi\nu\\ T = 2\pi/\omega = 1/\nu\\ \nu = 1/T = \omega/(2\pi)\end{cases}$$

We measure the period T or the frequency ν and calculate ω. The transition between spatial and temporal parts is done by the relations :

$$\begin{cases}\omega = kv_\varphi = 2\pi v_\varphi/\lambda = 2\pi\sigma v_\varphi\\ k = \omega/v_\varphi = 2\pi/(Tv_\varphi) = 2\pi\nu/v_\varphi\end{cases}$$

C.4 Harmonic Oscillator

Consider a free harmonic oscillator with one degree of freedom, for example a mass attached to a spring of constant stiffness k, oscillating in the linear part of the spring. The time equation of the mass

$$x(t) = x_{max} \cos(\omega t + \varphi_0)$$

is of the type of equation (3), where the function ψ is the elongation x. The restoring force model of the spring, $-\mathrm{k}x\mathbf{i}$, derives from the potential energy $\frac{1}{2}\mathrm{k}x^2$:

$$-\mathrm{k}x\mathbf{i} = -\,\mathbf{grad}\left(\tfrac{1}{2}\mathrm{k}x^2\right)$$

The mechanical energy E is the sum of potential and kinetic energy, the latter being zero at maximum elongation x_{max}, we have :

$$E = \tfrac{1}{2}\mathrm{k}x_{max}^2$$

The mechanical energy is therefore proportional to the square of the maximum amplitude of the mass motion.

C.5 Wave Function

To link the corpuscular and wave representations, we associate the energy of the wave ψ, that is, the square of its maximum amplitude ψ_{max}, to the probability of presence of the corpuscle. ψ is then called the *associated wave function* to the corpuscle. At a given point the probability of presence is zero, we must integrate over a domain of space to obtain a non-zero value. The square of the maximum amplitude ψ_{max} of the associated wave is therefore the volumetric probability density of presence of the corpuscle,

$$\psi_{max}^2 = \frac{dP}{dv}$$

so that the probability of presence of the corpuscle in the volume V is written :

$$P(V) = \int_V \psi_{max}^2, dv$$

When integrated over the entire volume that the quanton can occupy, the probability of presence of the latter must be equal to one :

$$\int_{V_{total}} \psi_{max}^2, dv = 1$$

We will see that the cosine function cannot serve as an associated wave function. Let the wave function be given by relation (3) :

$$\psi(x,t) = \psi_{max} \cos(kx - \omega t)$$

With the de Broglie relation (1b), we have :

$$\psi(x,t) = \psi_{max} \cos\left(\frac{p}{\hbar}, x - \omega t\right)$$

At time $t_0 = 0$,

$$\psi(x,t_0) = \psi_{max} \cos\left(\frac{p}{\hbar}, x\right)$$

the wave function cancels at points of abscissa x_n such that,

$$\forall n \in \mathbb{Z}, \qquad \frac{p}{\hbar}x_n = \frac{\pi}{2} + n\pi$$
$$x_n = \frac{h}{2p} + \frac{nh}{p}$$

and the distance between two points for which the wave function is zero is written :

$$x_{n+1} - x_n = \frac{h}{p}$$

If the observer is in a frame moving at speed v, the momentum of the quanton becomes $p' = p - m\text{v}$, and the points for which the wave function is zero are such that :

$$\forall n \in \mathbb{Z}, \qquad \frac{p'}{\hbar}x_n = \frac{\pi}{2} + n\pi$$
$$x_n = \frac{h}{2(p - m\text{v})} + \frac{nh}{p - m\text{v}}$$

The distance between two points for which the wave function is zero is written :

$$x_{n+1} - x_n = \frac{h}{p - m\text{v}}$$

This is impossible because the distance between two points is an invariant in non-relativistic physics.

D Complex Notation

We take advantage of the complex notation for circular functions. It is simpler to differentiate or integrate the exponential function because it does not change form. We therefore make the transition

$$\cos(kx - \omega t) \to e^{i(kx-\omega t)}$$

and we keep only the real part of the result. This method is justified in appendix I.2. Assuming the phase at the origin is zero, the periodic wave model (4) p. 6 is written

$$\psi(x,t) = \psi_{max}e^{i(kx-\omega t)} \tag{4}$$

and we have :

$$|\psi(x,t)|^2 = \psi_{max}^2$$

It is now the square of the modulus of the associated wave that represents the volumetric probability density of presence of the quanton :

$$|\psi(x,t)|^2 = \frac{dP}{dv}$$

$$\boxed{P(V) = \int_V |\psi(x,t)|^2\, dv}$$

When the modulus squared of ψ is integrated over the entire volume that the quanton can occupy, the probability of presence of the latter must be equal to one.

$$\int_{V_{total}} |\psi(x,t)|^2\, dv = 1$$

ψ is called the amplitude of probability density of presence of the quanton. The problem encountered with the cosine wave function remains present with the associated complex form :

$$\begin{aligned}\psi(x,t) &= \psi_{max} e^{i(kx-\omega t)} \\ &= \psi_{max} e^{i(\frac{P}{\hbar}x-\omega t)}\end{aligned}$$

At time $t_0 = 0$, the wave function is written :

$$\psi(x,t_0) = \psi_{max} e^{i\frac{P}{\hbar}x}$$

It does not cancel at any point but it equals ψ_{max} at points of abscissas x_n such that :

$$\begin{aligned}\forall n \in \mathbb{Z}, \qquad \frac{p}{\hbar} x_n &= 2\pi n \\ x_n &= \frac{2nh}{p}\end{aligned}$$

If the observer is in a frame moving at speed v, the momentum of the quanton becomes $p' = p - m\text{v}$, and the points for which the wave function equals ψ_{max} are such that :

$$\begin{aligned}\forall n \in \mathbb{Z}, \qquad \frac{p'}{\hbar} x_n &= 2\pi n \\ x_n &= \frac{2nh}{p - m\text{v}}\end{aligned}$$

The distance between two points for which the wave function equals ψ_{max} is written,

$$x_{n+1} - x_n = \frac{2h}{p - m\text{v}}$$

which is impossible, it cannot depend on the observer's speed.

E The Harmonic Plane Wave

Consider a gas contained in a parallelepiped enclosure. If one of the enclosure walls oscillates parallel to itself, a plane pressure wave is created in the gas. A wave plane is defined as a plane in which the pressure is equal at every point in that plane. Let O be the center of a reference

frame, and let P be any point in this wave plane. Let $\mathbf{r}$ be the position vector $\mathbf{OP}$, and $\mathbf{s}$ the unit velocity vector of direction and sense that of the propagation of the wave plane :

$$\mathbf{s} = \mathbf{v}/|\mathbf{v}|$$

The wave plane is defined by the set of points P such that at a fixed time t :

$$\mathbf{r}(P) \cdot \mathbf{s} = D(t)$$

It is the projection of $\mathbf{r}$ onto $\mathbf{s}$, where the time function $D(t)$ is the distance from point O to the wave plane at time t, as shown in figure 1.

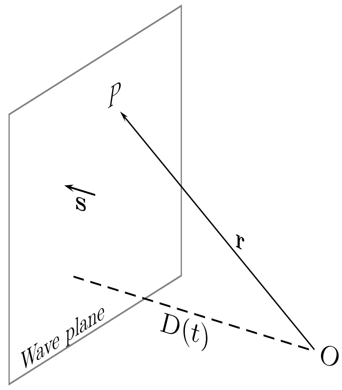

FIG 1. Wave plane

We can now generalize the wave equation (4) p. 6 to a plane wave :

$$\psi(\mathbf{r},t) = \psi_{max} e^{i(k\mathbf{r}\cdot\mathbf{s}-\omega t)}$$

For the confined gas example, we can verify that when located in the wave plane, that is, when the dot product $\mathbf{r}\cdot\mathbf{s}$ is constant, the pressure ψ is constant in space, and depends only on time. From the wave number defined in § C.1 p. 2, we define the *wave vector* by

$$\mathbf{k} \stackrel{\text{def}}{=} k\mathbf{s}$$

and the plane wave is written :

$$\boxed{\psi(\mathbf{r},t) = \psi_{max} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}} \tag{5}$$

F The Wave Packet

The simplest model of periodic plane wave is given by equation (5). Its frequency $\nu = \omega/(2\pi)$ being constant, this wave is said to be *monochromatic* (the wave has only one frequency, thus one color). This wave is modeled by a real cosine function plus an imaginary sine function, both having infinite spatial extension. It is therefore impossible to associate it with a particle localized in space, of finite spatial extension, and moreover we have seen that it cannot constitute an associated wave function. By adding several monochromatic plane waves of neighboring frequencies (or neighboring wave numbers since $k = 2\pi\nu/v_\varphi$) we observe the beating phenomenon. Amplitude maxima appear where these waves interfere constructively, and amplitude minima where they interfere destructively. The more waves we superimpose, the more the maxima space out and form «wave packets», and the more the amplitude between the maxima tends to zero (see appendix I.3 p. 13). Thanks to an integral over $\mathbf{k}$ in a domain from $\mathbf{k}_0 - \Delta\mathbf{k}$ to $\mathbf{k}_0 + \Delta\mathbf{k}$, we superimpose an infinity of plane waves all having a different wave vector :

$$\boxed{\psi(\mathbf{r},t) = \psi_{max} \int^{\mathbf{k}0+\Delta\mathbf{k}} \mathbf{k}0 - \Delta\mathbf{k} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} d\mathbf{k}} \tag{6}$$

There is then only one wave packet left, the others being sent to infinity. This wave packet will be our wave model of the quanton.

G Group Velocity

Let's find the expression for the velocity of the wave packet. We assume that ω varies as a function of k, not for the same wave since these are constants, but when passing from one wave to another in the wave packet. When this is the case

$$\omega = \omega(k)$$

and the medium is said to be *dispersive*. The wave function associated with the quanton (6) p. 8 becomes

$$\boxed{\psi(\mathbf{r},t) = \psi max \int^{\mathbf{k0}+\Delta\mathbf{k}} \mathbf{k0} - \Delta\mathbf{k} e^{i[\mathbf{k}\cdot\mathbf{r}-\omega(k)t]} d\mathbf{k}}$$

In a domain around k_0 sufficiently small, perform the first-order limited expansion of the pulsation in the vicinity of k_0 :

$$\begin{aligned}\omega(k) &= \omega_0 + \frac{d\omega}{dk}(k-k_0)\\ &= \omega_0 + \frac{d\omega}{dk}, k - \frac{d\omega}{dk}, k_0\end{aligned}$$

For wave propagation along the abscissa axis

$$\mathbf{k}\cdot\mathbf{r} = kx$$

and

$$\begin{aligned}kx - \omega t &= kx - \omega_0 t - \frac{d\omega}{dk}, kt + \frac{d\omega}{dk}, k_0 t\\ &= k_0 x - \omega_0 t - \left(\frac{d\omega}{dk}, t - x\right)(k-k_0)\end{aligned}$$

The wave function associated with the quanton is then written

$$\psi(x,t) = \psi max \int^{k_0+\Delta k} k_0 - \Delta k e^{i\left[k_0 x - \omega_0 t - \left(\frac{d\omega}{dk}, t - x\right)(k-k_0)\right]} dk$$

where Δk is small to stay in the vicinity of k_0.

$$\begin{aligned}\psi(x,t) &= \psi_{max} e^{i(k_0 x - \omega_0 t)} \int_{k_0-\Delta k}^{k_0+\Delta k} e^{i\left(x - \frac{d\omega}{dk}\,t\right)(k-k_0)} d(k-k_0)\\ &= \psi_{max} e^{i(k_0 x - \omega_0 t)} \left[\frac{e^{i\left(x-\frac{d\omega}{dk}\,t\right)(k-k_0)}}{i\left(x - \frac{d\omega}{dk}\,t\right)}\right]_{k_0-\Delta k}^{k_0+\Delta k}\\ &= \psi_{max} e^{i(k_0 x - \omega_0 t)} \times \frac{e^{i\left(x-\frac{d\omega}{dk}\,t\right)\Delta k} - e^{i\left(x-\frac{d\omega}{dk}\,t\right)\Delta k}}{i\left(x - \frac{d\omega}{dk}\,t\right)}\\ &= \psi_{max} e^{i(k_0 x - \omega_0 t)} \times \frac{2i \sin\left[\left(x - \frac{d\omega}{dk}\,t\right)\Delta k\right]}{i\left(x - \frac{d\omega}{dk}\,t\right)}\\ &= 2\psi_{max}\Delta k \frac{\sin\left[\left(x - \frac{d\omega}{dk}\,t\right)\Delta k\right]}{\left(x - \frac{d\omega}{dk}\,t\right)\Delta k} \times e^{i(k_0 x - \omega_0 t)}\\ &= 2\psi_{max}\Delta k \,\mathrm{sinc}\left[\left(x - \frac{d\omega}{dk}\,t\right)\Delta k\right] \times e^{i(k_0 x - \omega_0 t)}\end{aligned}$$

We recover the phase term $\exp[i(k_0 x - \omega_0 t)]$. The other terms form the amplitude of the wave. The amplitude variation propagates at the speed :

$$\boxed{v_g = \frac{d\omega}{dk}}$$

called *group velocity*. We then have :

$$\begin{aligned} v_g &= \frac{d(kv_\varphi)}{dk} \\ &= v_\varphi + k, \frac{dv_\varphi}{dk} \end{aligned}$$

where dv_φ/dk is the variation of v_φ as a function of k when passing from one wave to another. The group velocity is limited to the speed of light because it is the maximum speed of the quanton when modeled as a corpuscle. On the other hand, each monochromatic wave used to construct the wave packet, taken individually, does not model anything physical, which is why their phase velocity v_φ is not limited to the speed of light (see appendix I.4 p. 14).

H Dynamics of the Wave Packet

In the general case, the amplitude ψ_{max} of each wave is a function $f(\mathbf{k})$ of the wave vector, we have as the wave function associated with the quanton :

$$\boxed{\psi(\mathbf{r}, t) = \int^{\mathbf{k}0+\Delta\mathbf{k}} \mathbf{k}_0 - \Delta\mathbf{k} f(\mathbf{k}) e^{i[\mathbf{k}\cdot\mathbf{r} - \omega(k)t]} d\mathbf{k}}$$

We seek the wave equation of which the wave function $\psi(\mathbf{r}, t)$ is a solution. Using the transition relations (1) p. 1, we rewrite the wave function with the variables $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, which brings it into the domain of quantum mechanics by introducing the quantum of action :

$$\psi(\mathbf{r}, t) = \int f(\mathbf{k}) e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{k}$$

To build the differential equation, derive the wave function with respect to time :

$$\frac{\partial}{\partial t} \int f(\mathbf{k}) e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{k} = -i, \frac{E}{\hbar} \int f(\mathbf{k}) e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{k}$$

We see that we have the following equivalence between operators,

$$\frac{\partial}{\partial t} \quad \equiv \quad \times(-i)\frac{E}{\hbar}$$

that is,

$$\times E \quad \equiv \quad \times i\hbar \frac{\partial}{\partial t} \tag{7}$$

Multiplying the wave function by E is equivalent to deriving the wave function with respect to time and multiplying the result by $i\hbar$. Similarly, by deriving the wave function with respect to space,

$$\frac{\partial}{\partial \mathbf{r}} \int f(\mathbf{k}) e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{k} = i, \frac{\mathbf{p}}{\hbar} \int f(\mathbf{k}) e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{k}$$

we have the equivalence between operators :

$$\times\mathbf{p} \quad \equiv \quad \times(-i)\hbar, \frac{\partial}{\partial \mathbf{r}}$$

By deriving a second time with respect to space,

$$\frac{\partial^2}{\partial \mathbf{r}^2}\int f(\mathbf{k})e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)}d\mathbf{k} = -\frac{\mathbf{p}^2}{\hbar^2}\int f(\mathbf{k})e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)}d\mathbf{k}$$

we have :

$$\times\mathbf{p}^2 \quad \equiv \quad \times(-)\hbar^2\frac{\partial^2}{\partial \mathbf{r}^2} \tag{8}$$

Now that we have these equivalences between operators, multiply the conservation of energy equation (2) p. 2 by $\psi(\mathbf{r},t)$:

$$E_m\psi(\mathbf{r},t) = \frac{\mathbf{p}^2}{2m},\psi(\mathbf{r},t) + V(\mathbf{r},t),\psi(\mathbf{r},t)$$

We assume equality between the mechanical energy E_m of the corpuscle, and the energy E of the associated wave, and we replace the operators with their equivalents (7) and (8) :

$$i\hbar,\frac{\partial}{\partial t},\psi(\mathbf{r},t) = -\hbar^2\frac{\partial^2}{\partial \mathbf{r}^2},\psi(\mathbf{r},t) + V(\mathbf{r},t),\psi(\mathbf{r},t)$$

The multiplication by $V(\mathbf{r},t)$ also being an operator, we group the two operators on the right-hand side to form only one, and we obtain Schrödinger's equation :

$$\boxed{i\hbar,\frac{\partial}{\partial t},\psi(\mathbf{r},t) = \left[-\hbar^2\frac{\partial^2}{\partial \mathbf{r}^2} + V(\mathbf{r},t)\right]\psi(\mathbf{r},t)}$$

Definition H.1. We call *Hamiltonian*[a] and denote $\hat{H}$, the operator,

$$\hat{H} \stackrel{\text{def}}{=} -\hbar^2\frac{\partial^2}{\partial \mathbf{r}^2} + V(\mathbf{r},t)$$

[a]See Analytical Mechanics.pdf

so that Schrödinger's equation is also written,

$$\boxed{\hat{H}\left[\psi(\mathbf{r},t)\right] = E,\psi(\mathbf{r},t)}$$

where E is the mechanical energy.

I Appendices

I.1 Propagation Equation

Let $\psi = f(x)$ be any function of x. If a is a positive constant, $\psi = f(x-a)$ represents the same function translated by a in the direction of increasing x since the value $f(0)$ that was at $x=0$ is found at $x=a$. Now introduce time t to describe a «pulse» that propagates without deformation at speed v along the increasing x axis. It is represented by a function of variables x and t of the form :

$$f(x,t) = f(x-vt)$$

$f(x+vt)$ of course represents propagation along decreasing x.

I.2 Complex Notation

According to De Moivre's formula[3],

$$e^{i(kx-\omega t)} = \cos(kx-\omega t) + i\sin(kx-\omega t)$$

so :

$$\begin{aligned}\cos(kx-\omega t) &= \mathcal{R}e\{e^{i(kx-\omega t)}\}\\ \sin(kx-\omega t) &= \mathcal{I}m\{e^{i(kx-\omega t)}\}\end{aligned}$$

Denoting a the real part, and b the imaginary part, we must justify the transition $a \to a+ib$ and the return by taking the real part of the result. Let a be a wave function and a' another wave function:

I.2.1 Addition of Two Wave Functions

When adding two complex numbers $a+ib$ and $a'+ib'$ complexly, the real parts add, as do the imaginary parts, so that the sum of the real parts equals the real part of the sum :

$$\begin{aligned}a+a' &= \mathcal{R}e(a+a') + i(b+b')\\ &= \mathcal{R}e(a+ib) + (a'+ib')\end{aligned}$$

Similarly for the imaginary parts :

$$\begin{aligned}b+b' &= \mathcal{I}m(a+a') + i(b+b')\\ &= \mathcal{I}m(a+ib) + (a'+ib')\end{aligned}$$

In other words, we can perform the transition $a \to a+ib$, add, then take the real part of the result.

Remark. There is no formula for complex multiplication :

$$a \times a' \neq \mathcal{R}e(a+ib) \times (a'+ib')$$

but we will not need to multiply wave functions.

I.2.2 Derivative and Integral of a Wave Function

If a, b, a', and b' are functions of the parameter t, then the derivative of the real part equals the real part of the derivative :

$$\begin{aligned}\frac{da}{dt} &= \mathcal{R}e\left\{\frac{da}{dt} + i\frac{db}{dt}\right\}\\ &= \mathcal{R}e\left\{\frac{d}{dt}(a+ib)\right\}\end{aligned}$$

[3]See Trigonometry.pdf

Similarly for integration:

$$\int a dt = \mathcal{R}e \left\{ \int a dt + i \int b dt \right\}$$
$$= \mathcal{R}e \left\{ \int (a + ib) dt \right\}$$

In summary, we have shown that to add sinusoidal motions, we can add exponentials :

$$\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) = \mathcal{R}e\{e^{i(k_1 x - \omega_1 t)}\} + \mathcal{R}e\{e^{i(k_2 x - \omega_2 t)}\}$$
$$= \mathcal{R}e\{e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}\}$$

Similarly, we will differentiate and integrate exponentials :

$$\frac{d}{dt}\cos(kx - \omega t) = \frac{d}{dt}\mathcal{R}e\{e^{i(kx-\omega t)}\}$$
$$= \mathcal{R}e\left\{\frac{d}{dt}e^{i(kx-\omega t)}\right\}$$
$$\int \cos(kx - \omega t) dt = \int \mathcal{R}e\{e^{i(kx-\omega t)}\} dt$$
$$= \mathcal{R}e\left\{\int e^{i(kx-\omega t)} dt\right\}$$

I.3 Wave Packet

The following figures show the formation of isolated wave packets as we add sine functions of close frequencies :

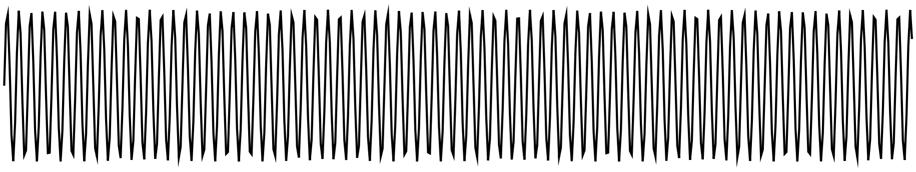

FIG 2. $\sin 40x$

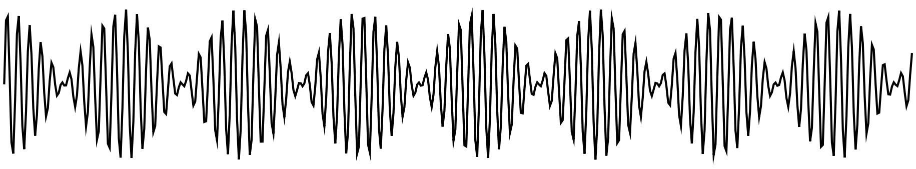

FIG 3. $\frac{1}{2}(\sin 40x + \sin 44x)$

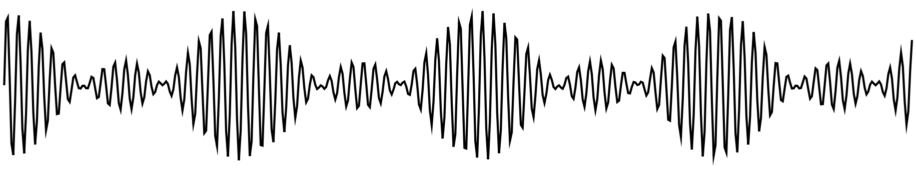

FIG 4. $\frac{1}{3}(\sin 40x + \sin 42x + \sin 44x)$

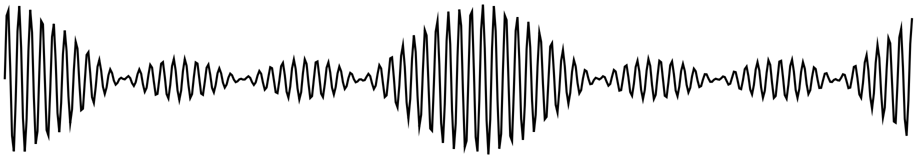

FIG 5. $\frac{1}{4}(\sin 40x + \sin 41x + \sin 42x + \sin 43x)$

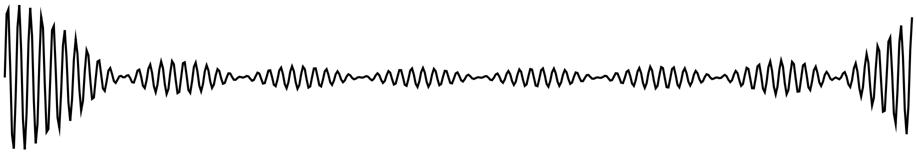

FIG 6. $\frac{1}{8}(\sin 40x + \sin 40.5x + \sin 41x + \sin 41.5x + \sin 42x + \sin 42.5x + \sin 43x + \sin 43.5x)$

I.4 Phase Velocity

Certain phenomena give the illusion of propagating faster than the speed of light. For example, a light chaser consisting of a set of aligned diodes that light up one after the other. If all diodes are lit at the same time, the apparent (fictitious) propagation speed of the signal is infinite. This is possible because the lighting of one diode is not subject to the lighting of the previous diode, there is no transport of information from one diode to the next, nor propagation of any signal. Similarly, a wave in a stadium could theoretically exceed the speed of light if each person decided to stand up at a precise moment, without caring about what the people on their sides are doing. There would then be no transport of information (from one person to another). If all people stand up at the same time, the wave speed is infinite. A last example is given by the breaking of a wave. If the wave breaks at once, the breaking speed is infinite. In our model, the monochromatic wave does not transport information, unlike the wave packet. Its speed is not limited to that of light.

Email address : o.castera@free.fr
Website : https://sciences-physiques.neocities.org