SQUARE ROOTS AND NTH ROOTS

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Abstract. Demonstration of Heron's sequence for calculating square roots using elementary operations of addition, subtraction, multiplication, and division. Then generalization of the method to nth roots.

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1 Numerical Example

We need to calculate $\sqrt{10}$. The calculator gives $\sqrt{10} = 3{,}162\,277\,66\ldots$ To start the sequence, we need an initial value. We choose 3 in this example because $\sqrt{10}$ is close to 3, but Heron's sequence converges regardless of the choice of initial value. A first approximation of $\sqrt{10}$ is given by the following calculation:

$$\frac{1}{2}\left(3 + \frac{10}{3}\right) = 3{,}166\,666\,6\ldots$$

To obtain more exact decimal places, we repeat the process. Note that it is unnecessary to keep all decimals in intermediate calculations:

$$\frac{1}{2}\left(3,17 + \frac{10}{3,17}\right) \approx 3{,}162\,287\,07\ldots$$

2 Proof of Heron's Sequence

The method used for calculating $\sqrt{10}$ is a sequence written as:

$$B_n = \frac{1}{2}\left(B_{n-1} + \frac{A}{B_{n-1}}\right)$$

in which $A = 10$ and $B_0 = 3$. Then we calculated:

$$B_1 = \frac{1}{2}\left(B_0 + \frac{A}{B_0}\right) = 3{,}166\,666\,6\ldots$$

$$B_2 = \frac{1}{2}\left(B_1 + \frac{A}{B_1}\right) \approx 3{,}162\,287\,07\ldots$$

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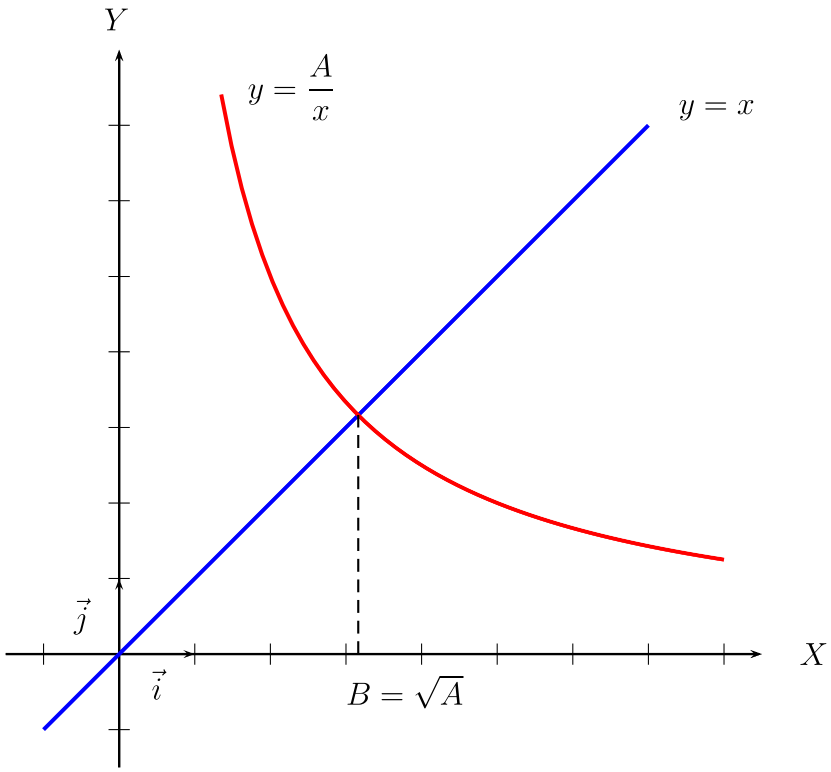
In what follows, parameters will be denoted in uppercase (A and B), while variables will be denoted in lowercase (x and y).

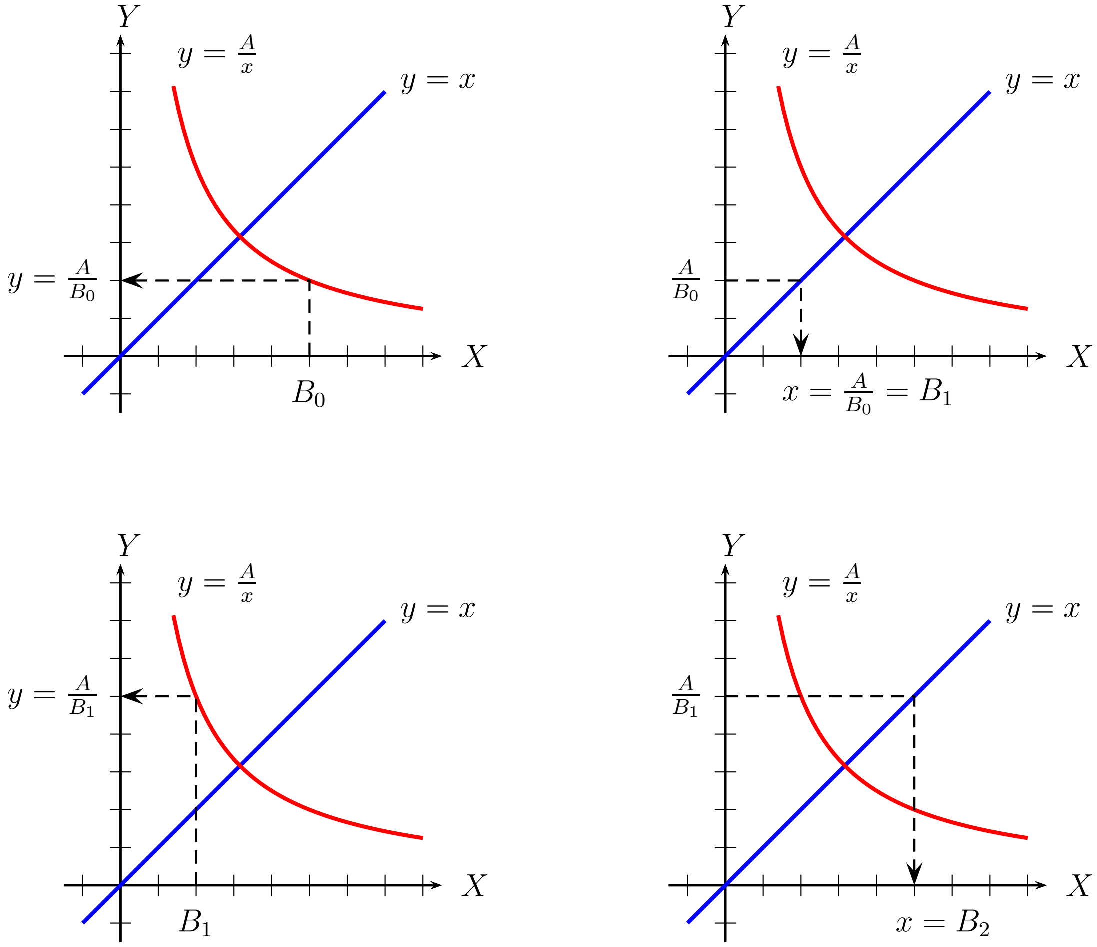
Let A be a number whose root B we seek, then:

$$B = \sqrt{A}$$
$$B^2 = A$$
$$B = \frac{A}{B}$$

To solve this equation, vary x until it is a solution to $x = A/x$. When x is a solution to the equation, it will have the value B. Therefore, if we plot the curves $y = x$ and $y = A/x$, they will intersect at $x = B$.



Let us see what happens if we use the sequence $B_n = A/B_{n-1}$ by fixing the initial value at B_0, with B_0 arbitrary and preferably close to $\sqrt{A}$. Once B_0 is fixed, we graphically find $B_1 = A/B_0$, then $B_2 = A/B_1$.



We observe that $B_2 = B_0$, which we can easily verify:

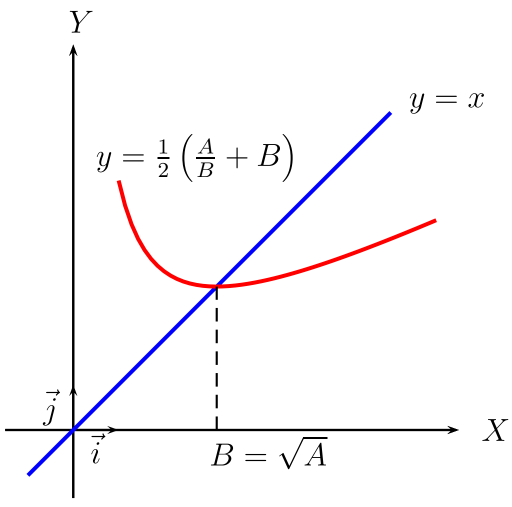
$$B_2 = \frac{A}{B_1} = \frac{A}{A/B_0} = B_0$$

This result shows that the sequence $B_n = A/B_{n-1}$ does not converge to $\sqrt{A}$. To make x tend toward $\sqrt{A}$, graphically, we need to approach the intersection of the two curves (here one of the curves is a line, but the solving method is used generally with two curves).

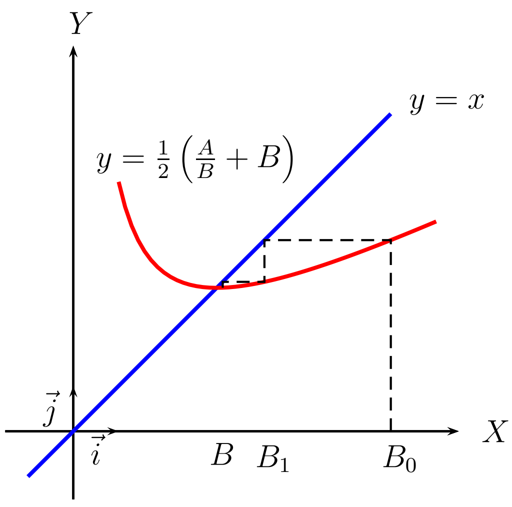
Let us rewrite the initial equation as follows:

$$\begin{aligned} B &= \frac{A}{B} \\ B + B &= \frac{A}{B} + B \\ 2B &= \frac{A}{B} + B \\ B &= \frac{1}{2}\left(\frac{A}{B} + B\right) \end{aligned}$$

We plot $y = x$ and $y = \frac{1}{2}\left(\frac{A}{B} + B\right)$. They intersect at $x = B$.



How does the new sequence $B_n = \frac{1}{2}\left(B_{n-1} + \frac{A}{B_{n-1}}\right)$ behave:



It converges quickly to $x = \sqrt{A}$. By examining the diagram above, we understand that the convergence condition of the sequence is that the curve $y = \frac{1}{2}\left(\frac{A}{x} + x\right)$ is minimal at $\sqrt{A}$. In fact, it suffices for it to be extremal at this point, whether a maximum or a minimum.

Let us verify that $\sqrt{A}$ is indeed an extremum of $y(x) = \frac{1}{2}\left(\frac{A}{x} + x\right)$

$$y'(x) = \frac{1}{2}\left(1 - \frac{A}{x^2}\right)$$

$$\frac{1}{2}\left(1 - \frac{A}{x^2}\right) = 0$$

$$\frac{A}{x^2} = 1$$

$$A = x^2$$

$$x = \pm\sqrt{A}$$

$\sqrt{A}$ is therefore indeed an extremum of $y(x) = \frac{1}{2}\left(\frac{A}{x} + x\right)$.

3 Generalization to Nth Roots

$$\begin{aligned} B^n &= A \\ B \times B^{n-1} &= A \\ B &= \frac{A}{B^{n-1}} \\ B + (n-1)B &= \frac{A}{B^{n-1}} + (n-1)B \\ nB &= \frac{A}{B^{n-1}} + (n-1)B \\ B &= \frac{1}{n}\left[\frac{A}{B^{n-1}} + (n-1)B\right] \end{aligned}$$

whence the sequence

$$B_m = \frac{1}{n}\left[(n-1)\,B_{m-1} + \frac{A}{B_{m-1}^{n-1}}\right]$$

Let us verify that $\sqrt[n]{A}$ is an extremum of $y(x) = \frac{1}{n}\left[(n-1)\,x + \frac{A}{x^{n-1}}\right]$

$$\begin{aligned} y'(x) &= \frac{1}{n}\left[(n-1) - (n-1)Ax^{-n}\right] \\ \frac{n-1}{n}\left(1 - Ax^{-n}\right) &= 0 \\ Ax^{-n} &= 1 \\ A &= x^n \\ x &= \pm\,\sqrt[n]{A} \end{aligned}$$

$\sqrt[n]{A}$ is therefore indeed an extremum of $y(x) = \frac{1}{n}\left[(n-1)\,x + \frac{A}{x^{n-1}}\right]$.

Remark 3.1. *In the case where A is negative, n must be odd, and we keep only the negative sign in the previous relation. For example, for $A = -27$, we have $3 = -\sqrt[3]{-27}$.*

4 Numerical Example

We need to calculate $\sqrt[5]{7}$. In this example, $A = 7$, $n = 5$, and we choose $B_0 = 1$.

$$
\begin{aligned}
B_0 &= 1 \\
B_1 &= \frac{1}{5}\left[4 \times 1 + \frac{7}{1^4}\right] = \frac{11}{5} = 2{,}2 \\
B_2 &= \frac{1}{5}\left[4 \times 2{,}2 + \frac{7}{2{,}2^4}\right] = 1{,}819\ 763\ 67\ldots \\
B_3 &= \frac{1}{5}\left[4 \times 1{,}82 + \frac{7}{1{,}82^4}\right] = 1{,}583\ 597\ 59\ldots \\
B_4 &= \frac{1}{5}\left[4 \times 1{,}58 + \frac{7}{1{,}58^4}\right] = 1{,}488\ 646\ 51\ldots \\
B_5 &= \frac{1}{5}\left[4 \times 1{,}49 + \frac{7}{1{,}49^4}\right] = 1{,}476\ 042\ 26\ldots
\end{aligned}
$$

The calculator gives $\sqrt[5]{7} = 1{,}475\ 773\ 161\ldots$

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