THE EQUATION OF TIME

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ABSTRACT. Over a year, the duration of the true solar day varies between 23 h 59 min 39 s and 24 h 0 min 30 s. These variations originate from the combination of two causes: the eccentricity of Earth's orbit and the obliquity of the ecliptic.

CONTENTS

1 Introduction

Throughout the year, the Sun does not pass at regular hours at its highest point in the sky for a stationary terrestrial observer. Sometimes ahead, sometimes behind a mean time, indicated by our clocks. The equation of time is the difference between this mean time and the true time obtained by observing the Sun's path:

$$\text{Equation of time} = \text{mean time (clocks) - true time (Sun)}$$

The extreme values of the difference are +14 min on February 11, the Sun is behind the clock, and −16 min on November 3, the Sun is ahead of the clock. The equation of time cancels out 4 times a year: on April 15, June 13, September 1, and December 25. It is around the winter solstice that the true solar days are the longest, and around the autumn equinox that they are the shortest. We will study the two causes behind the equation of time. The Earth rotates on itself around its rotation axis, noted North-South, which defines the true or astronomical North and South, distinct from the magnetic North and South indicated by a compass. This rotation occurs in the prograde direction (counterclockwise or trigonometric), viewed from the North. The North Pole is also called the boreal pole, and the South Pole, the austral pole. We will consider the Earth to be perfectly spherical, neglecting the flattening at the poles due to its rotation, its aspherical shape remnant of its formation, as well as the various reliefs.

Definition 1.1. *Meridian Plane*
The meridian plane is the plane containing the vertical of a location (in green on figure 1) and the Earth's rotation axis.

Definition 1.2. *Meridian*
Circle defined by the intersection of the meridian plane and the Earth's surface, drawn in gray on figure 1.

Date: December 11, 2025.

Definition 1.3. *Equatorial Plane*
Plane perpendicular to the rotation axis and passing through the Earth's center of gravity.

Definition 1.4. *Terrestrial Equator*
Great circle resulting from the intersection of the Earth's surface and the equatorial plane, drawn in blue on figure 1.

Definition 1.5. *Parallel*
Small circle on the terrestrial sphere, parallel to the equator plane, drawn in red on figure 1.

Definition 1.6. *Latitude*
Angle, noted ϕ, that the vertical of a location makes with the equator plane. This angle varies between $0°$ at the equator and $90°$ at the poles. On figure 1, $\phi = 48°N$. A N or S is noted after the angle value to specify whether it is the Northern or Southern hemisphere.

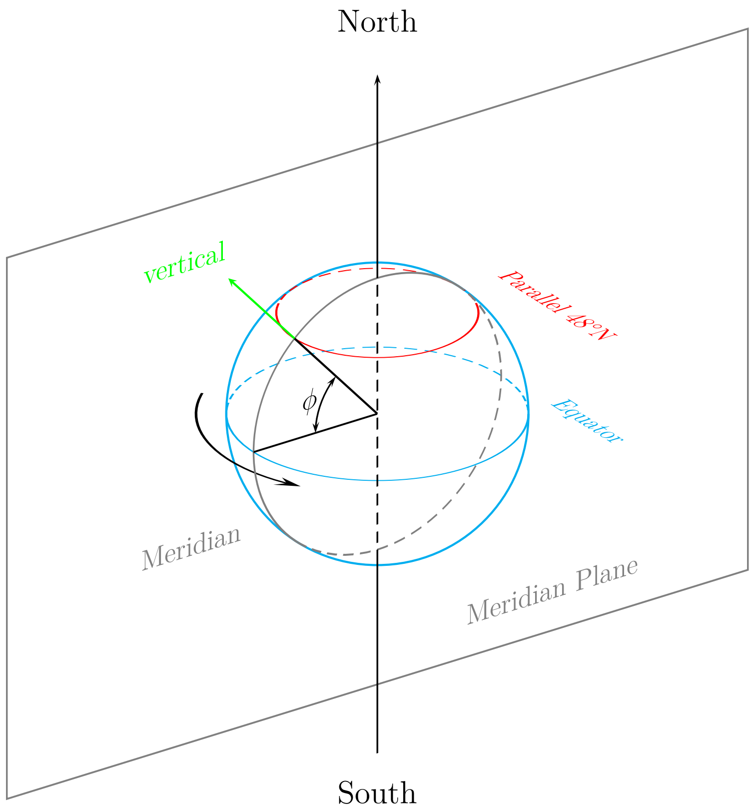

FIGURE 1. **A meridian and a parallel of the Earth**

Definition 1.7. *Solar Day*
Duration between two consecutive passages of the Sun at the same meridian.

Definition 1.8. *Mean Solar Day*
Average duration of the solar day determined over a very long time interval. It is fixed at 24 h *:*

$$1 \text{ mean solar day} = 86\,400\,\mathrm{s}$$

Definition 1.9. *True Noon*
It is true noon when the Sun is in the observer's meridian plane. The Sun then reaches its highest point in the observer's sky. It is also called upper passage or culmination.

Since the duration of the solar day is not constant, we fix the average duration of the solar day over a year at exactly 24 h. By dividing by 86400, we obtain the second of the mean solar day. The terrestrial and oceanic tides due to the gravitational interaction of the Earth with the Moon, the Sun, and the other planets in the solar system, cause the slowing of the Earth's rotation. The mean solar day losing 2.3 ms per century, the second of the mean solar day is longer than the second of the International System.

Theorem 1.1. *Kepler's First Law*
The Earth's orbit is an ellipse with the Sun at one of the foci.

More precisely, the Earth and the Sun rotate around their common center of gravity, which is almost coincident with that of the Sun. It is therefore the common center of gravity that is located at the focus of the ellipse described by the Earth.

Remark. *Kepler's three laws follow from mechanics and Newton's axiom of universal attraction inversely proportional to the square of the distances. Historically, Kepler first stated these laws using Tycho Brahe's observations.*

Definition 1.10. *Perihelion*
Point on the Earth's orbit closest to the Sun, occurring on January 3 in 2010.

Definition 1.11. *Aphelion*
Point on the Earth's orbit farthest from the Sun, occurring on July 6 in 2010.

Definition 1.12. *Apsides*
The perihelion and aphelion are called apsides of the orbit.

The major axis of the ellipse is therefore called the line of apsides. The Earth's orbit is not exactly an ellipse because its major axis rotates in the prograde direction by $11.461''$ per year. Nowadays, the Earth is at perihelion on January 3, a date to which 4 min 39 s must be added each year. This advance, in the prograde direction, of the Earth's perihelion and that of the other planets in the solar system, is explained by general relativity, and by the perturbations of the solar system's bodies among themselves.

Definition 1.13. *Eccentricity*
The eccentricity e of an elliptical orbit with semi-major axis a and semi-minor axis b (figure 2), is given by

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

The eccentricity of a circular orbit is therefore zero. The eccentricity of the Earth's orbit is equal to $0,016750$ on January 1, 1900, and decreases by $0,000043$ per century since that date.

Definition 1.14. *Ecliptic Plane*
Plane containing the average orbit of the Earth. In fact, it also contains the Sun and the Earth.

The other planets in the solar system are also in the ecliptic plane. By perturbing the trajectory of the other planets, Jupiter, the most massive planet in the solar system, has gradually imposed its plane of revolution on them. The successive orbits of the Moon form a complex twisted crown, due to its interaction with the Earth and the Sun. The plane of the Moon's orbit around the Earth is inclined by $5°$ on average on the ecliptic plane, and varies by $\pm 1°$. The Earth and the other planets in the solar system rotate in the prograde direction on themselves and around the Sun. The Sun also rotates on itself in the prograde direction in 25 days. This rotation of the celestial bodies in the prograde direction would originate from the conservation of angular momentum during the collapse of the gas cloud at the origin of the solar system.

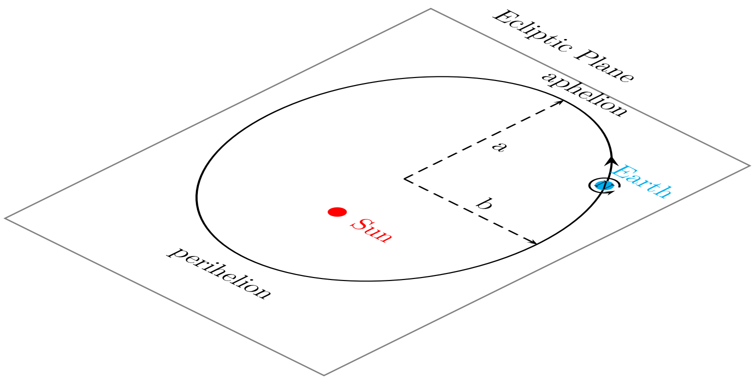

FIGURE 2. **Ecliptic plane**

Remark. *For readability reasons, the Sun, the Earth, and the axes of the ellipse will not be to scale in any of the figures in this document. For example, on figure 2, the eccentricity of $0,55$ is much higher than the real eccentricity.*

Definition 1.15. *Obliquity of the Ecliptic*
Angle of $23°26'$ that the Earth's rotation axis makes with the perpendicular to the ecliptic plane.

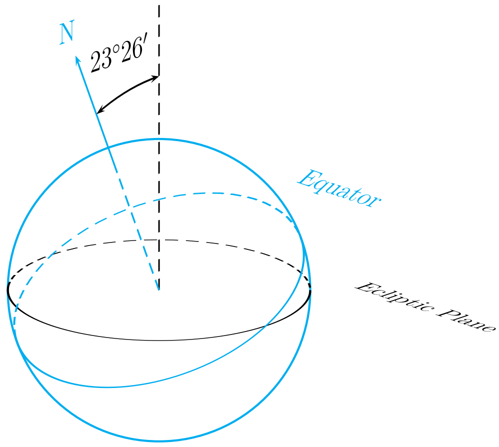

Figure 3. **Obliquity of the Ecliptic**

Definition 1.16. *Precession*
Revolution cone described by the Earth's rotation axis, in the retrograde direction (clockwise or inverse trigonometric), relative to the fixed stars. The cone is completed in 25 800 *years, which corresponds to* 50.256″ *per year.*

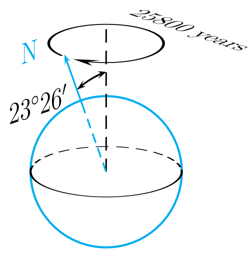

Figure 4. **Precession**

Remark. *The precession is due for two-thirds to the Moon's attraction and for one-third to the Sun's, on the Earth's equatorial bulge. In other words, if the Earth were perfectly spherical (if it did not rotate on itself), there would be no precession movement.*

Remark. *The so-called fixed stars are distant stars whose movement can often be considered negligible. To be fixed, an absolute reference frame would be needed, which we know has no reality. Usage has it that we speak of movement relative to the fixed stars although it would be more correct to speak of movement relative to the rest of the observable Universe.*

Due to the Earth's interaction with the other planets in the solar system, the obliquity oscillates, over a very long period, between two extreme values, 21°59′ and 24°36′. It is currently decreasing by 47.6″ per century.

Definition 1.17. *Nutation*
Quasi-sinusoidal movement with a period of 18.6 *years and maximum amplitude* 9.21″*, described by the Earth's rotation axis during its precession.*

Under the action of nutation alone, the Earth's rotation axis would describe a small elliptical cone, whose major axis of 18.42″ would be directed towards the perpendicular to the ecliptic passing through the Earth's center, and whose minor axis would measure 13.72″. The nutation is almost entirely due to the Moon's attraction. The Earth's rotation axis ultimately has a movement that is the result of precession and nutation. The poles describe a sort of sinusoid festooning this circle with 1390 scallops, each of which only nibbles away at the $1/9170^{th}$ part of its radius.

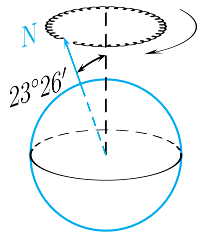

FIGURE 5. **Precession and Nutation**

On figure 5, 40 sinusoids are represented, with amplitude $1/20^{th}$ part of the radius.

Definition 1.18. *Solstice*
Each of the two periods of the year when the inequality in the duration of day and night is the greatest over the entire Earth.

When the Earth is at the solstices, the Earth's rotation axis is tilted *towards the Sun* by 23°26′.

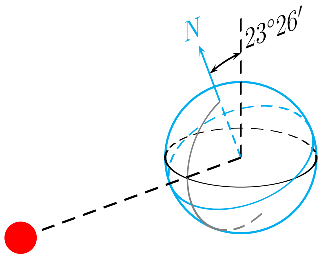

FIGURE 6. **Summer Solstice in the Northern Hemisphere**

Definition 1.19. *Equinox*
Each of the two periods of the year when the day has a duration equal to that of the night over the entire Earth.

When the Earth is at the equinoxes, its rotation axis is tilted by 23°26′ *perpendicularly* to the Earth-Sun line:

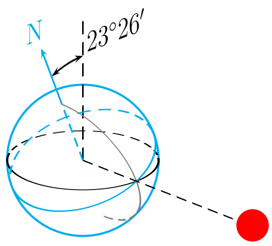

FIGURE 7. **Autumn Equinox in the Northern Hemisphere**

We will approximate that over a year, the Earth's rotation axis remains fixed relative to the rest of the Universe. We will therefore neglect the precession and nutation movements, as well as the advance of the perihelion.

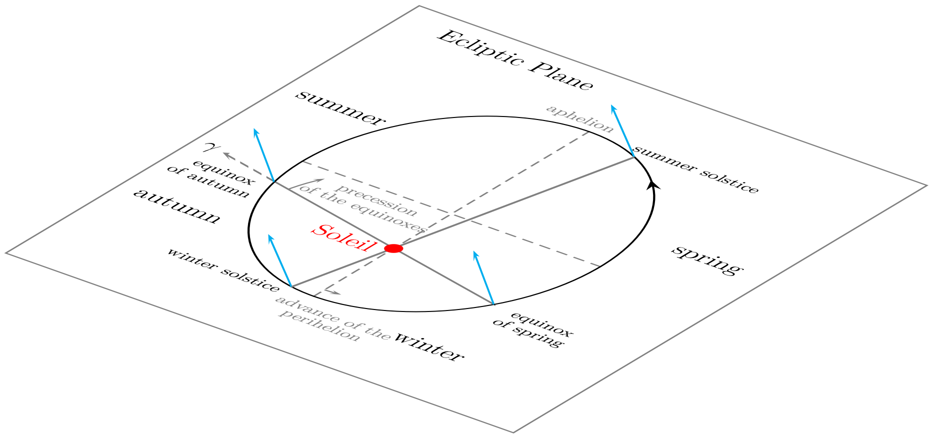

FIGURE 8. **The Earth's rotation axis is quasi-fixed**

Definition 1.20. *Equinoctial Points*
The equinoctial points γ and γ' are the intersection points of the equatorial plane and the ecliptic. They are also called the nodes of the ecliptic.

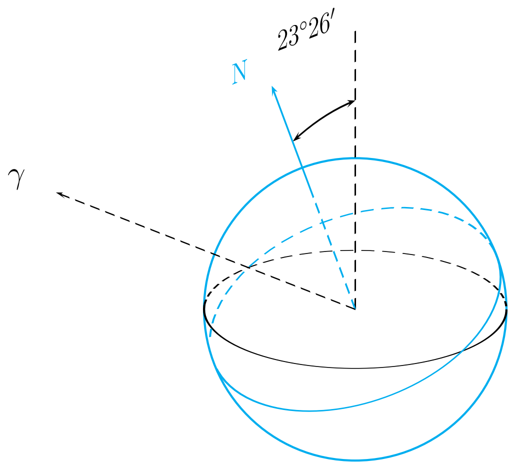

FIGURE 9. **Gamma point**

The point γ, also called the *vernal* point, can be considered as a fictitious star, which, viewed from Earth on the day of the spring equinox, would be located behind the Sun. See figure 8 on which the Earth's rotation axis, represented in blue, is outside the ecliptic plane.

Definition 1.21. *Sidereal Year*
Time interval separating two consecutive conjunctions of the Sun and a fixed star. In other words, when the Sun and the star are in the plane containing the perpendicular to the ecliptic passing through the Earth's center.

The sidereal year has 365.256 360 mean solar days.

Remark. *Since sidereal time is relative to the vernal point, the sidereal year should have been defined relative to this point, not relative to fixed stars. The previous definition should be that of the stellar year.*

Definition 1.22. *Tropical Year*
Time interval separating two consecutive spring equinoxes.

The tropical year has 365.242 190 mean solar days. Due to the precession of the Earth's rotation axis, also called precession of the equinoxes, the tropical year lasts 19 min 59 s less than the sidereal year :

$$\begin{aligned}(365.256\,360 - 365.242\,190) \times 86\,400 &= 1199\,\mathrm{s}\\ &= 19\,\mathrm{min}\ 59\,\mathrm{s}\end{aligned}$$

Definition 1.23. *Anomalistic Year*
Time interval separating two consecutive passages of the Earth at perihelion.

The anomalistic year has 365.259 635 mean solar days. Due to the advance of the Earth's perihelion, the anomalistic year lasts 4 min 43 s more than the sidereal year :

$$\begin{aligned}(365.259\,635 - 365.256\,360) \times 86\,400 &= 283\,\mathrm{s}\\ &= 4\,\mathrm{min}\ 43\,\mathrm{s}\end{aligned}$$

2 Eccentricity of the Earth's Orbit

The Earth's rotation on itself is accompanied by an arc of revolution around the Sun, so that a rotation of the Earth on itself referring to the Sun does not have the same duration as a rotation referring to the fixed stars:

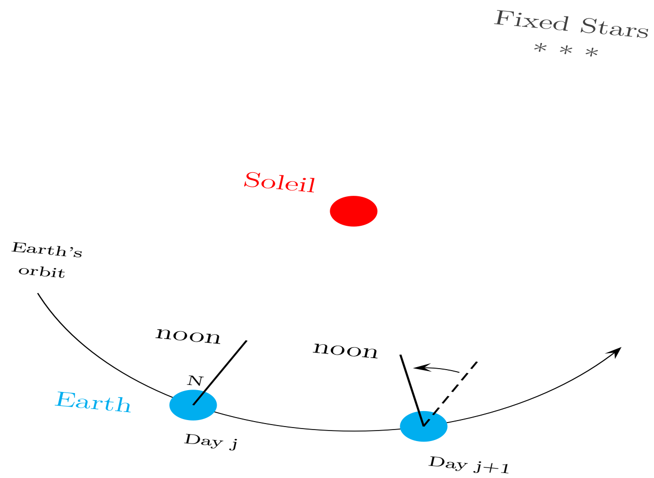

Figure 10. **Duration of the Day**

Definition 2.1. *Sidereal Day*
Duration between two consecutive passages of the vernal point at the meridian.

Unlike the sidereal year which is defined relative to fixed stars, the sidereal day is defined relative to the vernal point. The Earth rotating on itself in the same direction as around the Sun, prograde direction, the solar day is longer than the stellar day or the sidereal day. Over a tropical year there will be one more sidereal day than mean solar days:

$$365.242\,190 \textit{ mean solar days} = 366.242\,190 \textit{ sidereal days}$$

This relation gives us the precise duration of the sidereal day :

$$\begin{aligned}\frac{365.242\,190}{366.242\,190} \times 86\,400 &= 86\,164.0905\,\mathrm{s}\\ &= 23\,\mathrm{h}\ 56\,\mathrm{min}\ 4.0905\,\mathrm{s}\end{aligned}$$

in seconds of mean solar day.

Definition 2.2. *Stellar Day*
Duration between two consecutive passages of a fixed star at the meridian.

The precession causing the retrograde displacement of the point γ on the ecliptic, the stellar day is longer than the sidereal day and has a duration of 23 h 56 min 4.0997 s :

$$\begin{aligned}\frac{365.256\,360}{366.256\,360} \times 86\,400 &= 86\,164.0997\,\mathrm{s}\\ &= 23\,\mathrm{h}\ 56\,\mathrm{min}\ 4.0997\,\mathrm{s}\end{aligned}$$

Theorem 2.1. *Kepler's Second Law*
The Earth-Sun radius vector sweeps equal areas in equal times.

This second law implies that the Earth's speed varies on its orbit, that it is maximum at perihelion, and minimum at aphelion. At aphelion the solar day is equal to the sidereal day plus the time taken to cover the angle α, and at perihelion the solar day is equal to the sidereal day plus the time taken to cover the angle β (see figure 11).

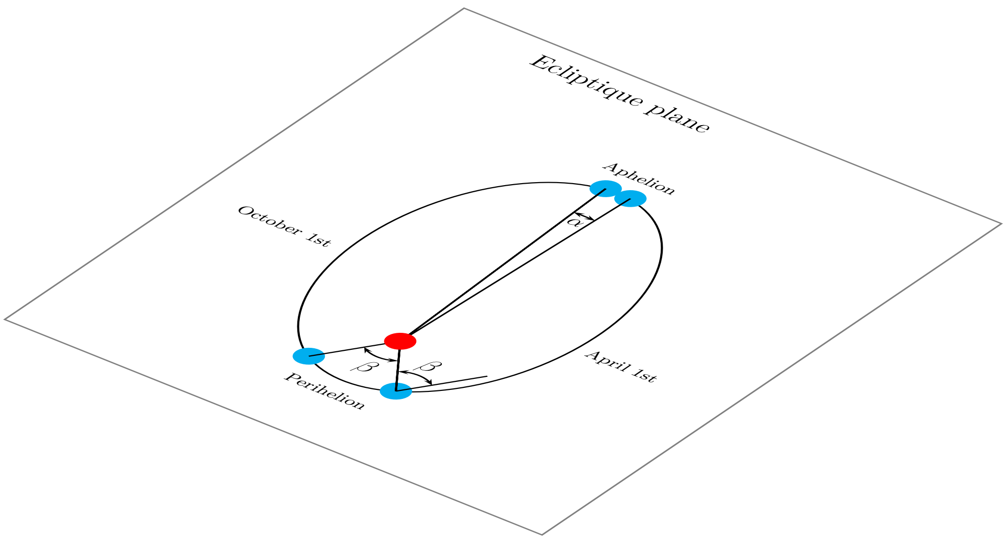

FIGURE 11. **Earth's Revolution around the Sun**

The areas swept being equal for equal time intervals, the angle β is greater than the angle α. If we consider only the component due to the eccentricity of the Earth's orbit, then : from October 1st to April 1st, the Sun accumulates delay on its passage at the meridian because the angle is large between two complete rotations relative to the Sun. On April 1st, its delay is maximum. From April 1st to October 1st the Sun catches up its delay, and on October 1st, its advance on the mean Sun indicated by our watches is maximum. Thus, the eccentricity of the Earth's orbit causes a quasi-sinusoidal variation of the duration of the solar day, with period the tropical year.

3 Obliquity of the Ecliptic

The Earth's orbit is assumed circular since the problem related to eccentricity was treated in the previous chapter. We first assume that the rotation axis is not tilted on the ecliptic. By symmetry, regardless of the time of year, true noon and mean noon coincide.

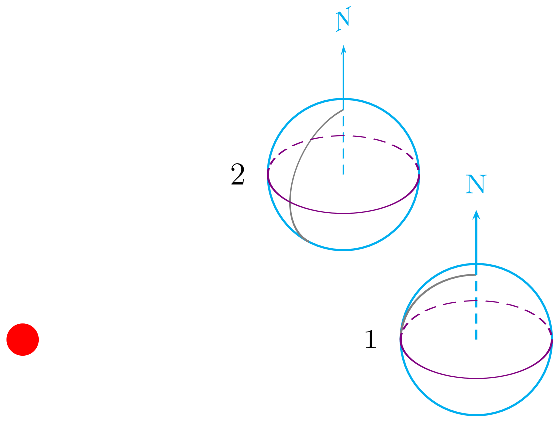

Figure 12. **True Noon and Mean Noon Coincide**

During the 24 h separating positions 1 and 2, the Earth performs one turn plus about $1/365.25^{th}$ of a turn, and the Sun is back at the zenith of the meridian shown in gray. In violet, the ecliptic plane coincides with the equatorial plane. Now let's tilt the Earth's rotation axis by performing a rotation of $23°26'$ around the vernal axis $(\gamma\gamma')$, as shown in figure 13. The ecliptic plane in red separates from the equatorial plane, in blue.

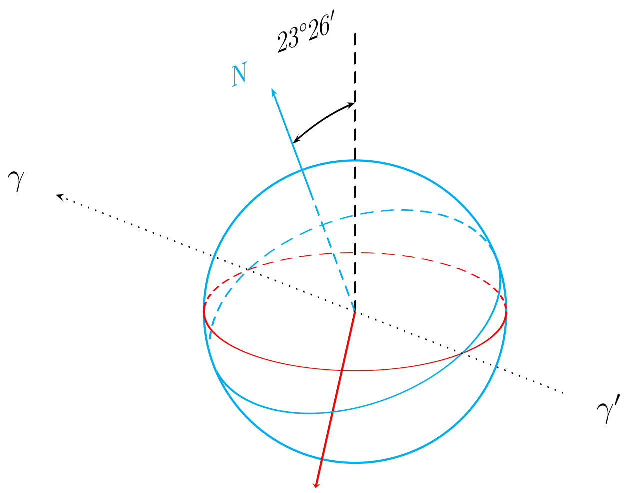

Figure 13. **Tilt on the Ecliptic Plane**

The red arrow indicates the direction of the Sun. The angle between the vernal axis and the direction of the Sun corresponds to the time of year, here it is between the summer solstice and the autumn equinox. Let's perform this rotation by placing ourselves in the geocentric frame (figure 14).

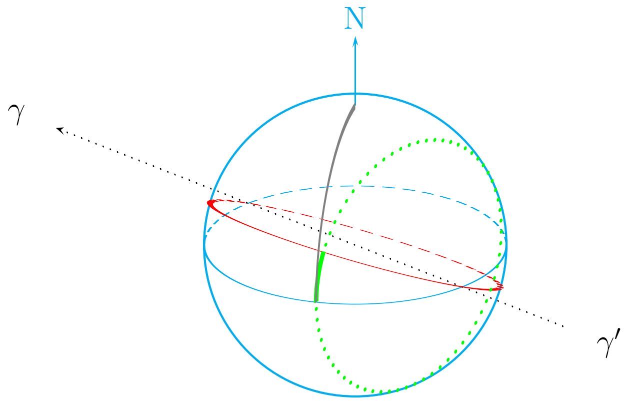

FIGURE 14. **In the Geocentric Frame**

For a person on the equator and on the gray meridian where clocks show noon, during this rotation the Sun describes an arc of small circle (in green) centered on the vernal axis. The Sun leaves the gray meridian on which it was, it is no longer true noon for the person. The Earth must rotate a little more for the Sun to return to the meridian: the Sun is delayed. At the equinoxes the Sun is on the vernal axis, the obliquity does not take it out of the meridian. The same goes for the solstices, since the obliquity makes it follow the meridian on which it is. The tilt of the Earth's rotation axis on the ecliptic therefore also causes a quasi-sinusoidal variation of the duration of the solar day, but with a period of 6 months.

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